44. *Probabilities on Inheritance in Consanguineous Families. XIII*

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X. Combinations through extreme consanguineous marriages, 2

1. Definitions of quantities

In the present chapter we shall supplement the results on special combinations previously postponed as an intermediate extreme case. We first attempt to determine the probability of mother-descendants combination of the form

$$(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)_{(1\nu;\,0)_t|1\nu}\equiv(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)_{1\nu_1;\,0|\dots|1\nu_t;\,0|1\nu_{t+1}},$$

the generation-numbers $\nu_r\,(1\leqq r\leqq t+1)$ being supposed greater than unity.

We shall collect previously in the present section the definition of several quantities which will become necessary for establishing general formulas.

We first define a quantity $\mathfrak{R}^{\backprime}$ by

$$\mathfrak{R}^{\backprime}(\xi_1\eta_1,\ \xi_2\eta_2)=2\mathfrak{R}^*(\xi_1\eta_1,\ \xi_2\eta_2)-\bar{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\ \xi_2\eta_2).$$

We next introduce two quantities $\mathfrak{S}^{\backprime}$ and $\mathfrak{S}^{\backprime\backprime}$ by means of

$$\begin{aligned}16\sum\mathfrak{S}^*(\alpha\beta;\ ab,\ cd)\varepsilon(ab,\ cd;\ \xi_1\eta_1)E(ab,\ cd;\ \xi_2\eta_2)\\=14\mathfrak{S}^*(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)-3S(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)-\mathfrak{S}^{\backprime}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)\\=12\mathfrak{S}^*(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)-2S(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)+\mathfrak{S}^{\backprime\backprime}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)\end{aligned}$$

and further two quantities $\mathfrak{T}^{\backprime}$ and $\mathfrak{T}^{\backprime\backprime}$ by means of

$$\begin{aligned}16\sum\mathfrak{T}^*(\alpha\beta;\ ab,\ cd)\varepsilon(ab,\ cd;\ \xi_1\eta_1)E(ab,\ cd;\ \xi_2\eta_2)\\=14\mathfrak{T}^*(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)-3T(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)-\mathfrak{T}^{\backprime}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)\\=12\mathfrak{T}^*(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)-2T(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)+\mathfrak{T}^{\backprime\backprime}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2).\end{aligned}$$

There then hold evidently the relations

$$\begin{aligned}\mathfrak{S}^*(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=\tfrac{1}{2}\{S(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)+\mathfrak{S}^{\backprime}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)\\+\mathfrak{S}^{\backprime\backprime}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)\},\\\mathfrak{T}^*(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=\tfrac{1}{2}\{T(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)+\mathfrak{T}^{\backprime}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)\\+\mathfrak{T}^{\backprime\backprime}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)\}.\end{aligned}$$

We finally introduce two quantities V^* and W^* by

$$V^*(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=V(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)-\bar{A}_{\xi_1\eta_1}Q(\alpha\beta;\ \xi_2\eta_2)-S(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2),$$

$$W^*(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=W(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)-2\bar{A}_{\xi_1\eta_1}Q(\alpha\beta;\ \xi_2\eta_2)-T(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2).$$

However, though we have not yet explicitly noticed, there holds a remarkable identity

$$2V(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)-W(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=2S(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)-T(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)$$

so that we may forsake the symbol V^*, since there holds a corresponding identity

$$V^*(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=\tfrac{1}{2}W^*(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2).$$

For our later purpose, rather than W^* itself, direct use will be made of a related quantity $W^{\backprime}$ defined by a relation

$$W^*(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=\tfrac{1}{2}\{W^{\backprime}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)+8\mathfrak{S}^{\backprime\backprime}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)\}.$$

The values of every quantity thus defined can be obtained by actual computation based on the respective definition while they will be omitted here for economy reason of space.[1]

2. Mother-descendants combination

We now deal with the probability $\kappa_{(1\nu;\,0)_t|1\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)$ with $\nu_r>1$ for $1\leqq r\leqq t+1$. It satisfies evidently a recurrence equation

$$\kappa_{(1\nu;0)_t|1\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=\sum\kappa_{(1\nu;\,0)_t}(\alpha\beta;\ ab,\ cd)\varepsilon(ab,\ cd;\ \xi_1\eta_1)\varepsilon_\nu(ab,\ cd;\ \xi_2\eta_2)$$

which may be regarded as *a system of linear difference equations*, unknowns being the $\kappa_{(1\nu;\,0)_t|1\nu}$'s and an independent variable being t, i.e. the number of interjacent consanguineous marriages. Initial condition for the system is given by a previously derived relation

$$\begin{aligned}\kappa_{1\nu_1}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)&=\bar{A}_{\xi_2\eta_2}\kappa(\alpha\beta;\ \xi_1\eta_1)+2^{-\nu_1}W(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)\\&=\bar{A}_{\xi_1\eta_1}\bar{A}_{\xi_2\eta_2}+\bar{A}_{\xi_2\eta_2}Q(\alpha\beta;\ \xi_1\eta_1)+2^{-\nu_1+1}\bar{A}_{\xi_1\eta_1}Q(\alpha\beta;\ \xi_2\eta_2)+2^{-\nu_1}T(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)\\&\quad+2^{-\nu_1+1}W^{\backprime}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)+2^{-\nu_1+2}\mathfrak{S}^{\backprime\backprime}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2).\end{aligned}$$

It can be shown that *the system is solved by the formula*

$$\begin{aligned}\kappa_{(1\nu;0)_t|1\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)&=[\bar{A}_1\bar{A}_2]_{t+1}\bar{A}_{\xi_1\eta_1}\bar{A}_{\xi_2\eta_2}+[\bar{A}_2Q(\mathbf{0};\mathbf{1})]_{t+1}\bar{A}_{\xi_2\eta_2}Q(\alpha\beta;\ \xi_1\eta_1)\\&\quad+[\bar{A}_1Q(\mathbf{0};\mathbf{2})]_{t+1}\bar{A}_{\xi_1\eta_1}Q(\alpha\beta;\ \xi_2\eta_2)\\&+[\bar{A}_1Q(\mathbf{1};\mathbf{2})]_{t+1}\bar{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\ \xi_2\eta_2)+[\bar{A}_2T(\mathbf{0};\mathbf{1})]_{t+1}\bar{A}_{\xi_2\eta_2}T(\alpha\beta;\ \xi_1\eta_1)\\&\quad+[\bar{A}_2S(\mathbf{0};\mathbf{1})]_{t+1}\bar{A}_{\xi_2\eta_2}S(\alpha\beta;\ \xi_1\eta_1)\\&+[\bar{A}_2R(\mathbf{1})]_{t+1}\bar{A}_{\xi_2\eta_2}R(\xi_1\eta_1)+[\mathfrak{R}^{\backprime}(\mathbf{1},\mathbf{2})]_{t+1}\mathfrak{R}^{\backprime}(\xi_1\eta_1,\ \xi_2\eta_2)\\&\quad+[T(\mathbf{0};\mathbf{1},\mathbf{2})]_{t+1}T(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)\\&+[S(\mathbf{0};\mathbf{1},\mathbf{2})]_{t+1}S(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)+[\mathfrak{T}^{\backprime}(\mathbf{0};\mathbf{1},\mathbf{2})]_{t+1}\mathfrak{T}^{\backprime}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)\\&\quad+[\mathfrak{T}^{\backprime\backprime}(\mathbf{0};\mathbf{1},\mathbf{2})]_{t+1}\mathfrak{T}^{\backprime\backprime}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)\\&+[W^{\backprime}(\mathbf{0};\mathbf{1},\mathbf{2})]_{t+1}W^{\backprime}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)+[\mathfrak{S}^{\backprime}(\mathbf{0};\mathbf{1},\mathbf{2})]_{t+1}\mathfrak{S}^{\backprime}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)\\&\quad+[\mathfrak{S}^{\backprime\backprime}(\mathbf{0};\mathbf{1},\mathbf{2})]_{t+1}\mathfrak{S}^{\backprime\backprime}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2),\end{aligned}$$

where all the coefficients $[\quad]_{t+1}$ *depend only on the generation-numbers concerned and are really expressed in the following form:*

$$[\bar{A}_1\bar{A}_2]_{t+1}=1,$$

$$[\bar{A}_2Q(\mathbf{0};\mathbf{1})]_{t+1}=\Lambda_t,$$

$$[\bar{A}_1Q(\mathbf{0};\mathbf{2})]_{t+1}=\Lambda_t2^{-\nu_{t+1}+1},$$

1) Full tables for these values will be listed in a forthcoming paper in Bull. Tokyo Inst. Tech. (1955).

$$[\bar{A}_1 Q(1;2)]_{t+1}=2^{-M_{t+1}}\sum_{p=0}^{[(t-1)/2]}2^{-2p+1}\sum_{q=1}^{t-2p}2^{M_q}\sum_{B^{(p)}_{q,t-2p}}\prod_{r=1}^{p}2^{\nu_{s_r}},$$

$$[\bar{A}_2 T(0;1)]_{t+1}=2^{-M_t}\sum_{p=0}^{[(t-1)/2]}2^{-2p}\sum_{B^{(p)}_{0,t-2p-1}}\prod_{r=1}^{p}2^{\nu_{s_r}},$$

$$[\bar{A}_2 S(0;1)]_{t+1}=2^{-M_t}\sum_{p=0}^{[t/2]-1}2^{-2p}\sum_{q=1}^{t-2p-1}\Lambda_q 2^{M_q}\sum_{B^{(p)}_{q,t-2p-1}}\prod_{r=1}^{p}2^{\nu_{s_r}},$$

$$[\bar{A}_2 R(1)]_{t+1}=2^{-M_t}\sum_{p=0}^{[t/2]-1}2^{-2p}\sum_{q=1}^{t-2p-1}2^{M_q}\sum_{B^{(p)}_{q,t-2p-1}}\prod_{r=1}^{p}2^{\nu_{s_r}},$$

$$[\mathfrak{R}'(1,2)]_{t+1}=2^{-M_{t+1}}\sum_{p=0}^{t-2}2^{-p+1}\sum_{q=1}^{t-p-1}2^{M_q}\Big\{1+\sum_{u=1}^{[p/2]}\sum_{B^{(u)}_{q,t-p-1}}\prod_{r=1}^{u}2^{\nu_{s_r}}\Big\},$$

$$[T(0;1,2)]_{t+1}=2^{-M_{t+1}}\sum_{p=0}^{[t/2]}2^{-2p}\sum_{B^{(p)}_{0,t-2p}}\prod_{r=1}^{p}2^{\nu_{s_r}},$$

$$[S(0;1,2)]_{t+1}=2^{-M_{t+1}}\sum_{p=0}^{[(t-1)/2]}2^{-2p+1}\sum_{q=1}^{t-2p}\Lambda_q 2^{M_q}\sum_{B^{(p)}_{q,t-2p}}\prod_{r=1}^{p}2^{\nu_{s_r}},$$

$$[\mathfrak{T}'(0;1,2)]_{t+1}=2^{-M_{t+1}}\sum_{p=0}^{t-1}2^{-p}\Big\{1+\sum_{u=1}^{[p/2]}\sum_{B^{(u)}_{0,t-p-1}}\prod_{r=1}^{u}2^{\nu_{s_r}}\Big\},$$

$$[\mathfrak{T}''(0;1,2)]_{t+1}=2^{-M_{t+1}}\Big\{(-4)^{-t+1}+\sum_{v=0}^{t-2}(-4)^{-v}\Big(\sum_{p=0}^{[(t-v-1)/2]}2^{-2p+1}\sum_{B^{(p)}_{0,t-v-2p-2}}\prod_{r=1}^{p}2^{\nu_{s_r}}$$
$$+2^{\nu_{t-v-1}}\sum_{p=1}^{[(t-v-1)/2]}2^{-2p}\sum_{B^{(p)}_{0,t-v-2p-1}}\prod_{r=1}^{p-1}2^{\nu_{s_r}}+\sum_{p=0}^{t-v-3}2^{-p+1}\Big(1+\sum_{u=1}^{[p/2]}\sum_{B^{(u)}_{0,t-v-p-3}}\prod_{r=1}^{u}2^{\nu_{s_r}}\Big)\Big)\Big\},$$

$$[W'(0;1,2)]_{t+1}=2^{-M_{t+1}-t-1}\sum_{q=0}^{t}\Lambda_q 2^{M_q+q},$$

$$[\mathfrak{S}'(0;1,2)]_{t+1}=2^{-M_{t+1}}\sum_{p=0}^{t-2}2^{-p+1}\sum_{q=1}^{t-p-1}\Lambda_q 2^{M_q}\Big\{1+\sum_{u=1}^{[p/2]}\sum_{B^{(u)}_{q,t-p-1}}\prod_{r=1}^{u}2^{\nu_{s_r}}\Big\},$$

$$[\mathfrak{S}''(0;1,2)]_{t+1}=2^{-M_{t+1}}\sum_{v=0}^{t}(-4)^{-v}\Big(2^{-t+v+2}+4\sum_{q=1}^{t-v}\Lambda_q 2^{M_q}$$
$$+\sum_{p=0}^{t-v-3}2^{-p}\sum_{q=1}^{t-v-p-2}\Lambda_q 2^{M_q}+2^{\nu_{t-v-1}}\sum_{p=0}^{[(t-v)/2]-2}2^{-2p-1}\sum_{q=1}^{t-v-2p-3}\Lambda_q 2^{M_q}\sum_{B^{(p)}_{q,t-v-2p-3}}\prod_{r=1}^{p}2^{\nu_{s_r}}$$
$$+\sum_{p=1}^{[(t-v-1)/2]-1}2^{-2p+2}\sum_{q=1}^{t-v-2p-2}\Lambda_q 2^{M_q}\sum_{B^{(p)}_{q,t-v-2p-2}}\prod_{r=1}^{p}2^{\nu_{s_r}}+\sum_{p=2}^{t-v-4}2^{-p}\sum_{q=1}^{t-v-p-3}\Lambda_q 2^{M_q}\sum_{u=1}^{[p/2]}\sum_{B^{(u)}_{q,t-v-p-3}}\prod_{r=1}^{u}2^{\nu_{s_r}}\Big).$$

Here we put, for the sake of brevity,

$$\Lambda_q=\prod_{s=1}^{q}(2^{-1}+2^{-\nu_s})\qquad\text{and}\qquad M_q=\sum_{s=1}^{q}\nu_s.$$

Further the symbol $[\tau]$ standing over $\sum$ denotes the Gauss' one which represents the largest integer not exceeding τ. On the other hand, the symbol $B^{(u)}_{q,w}$ standing below $\sum$ denotes a u-dimensional range in $(s_1,\dots,s_u)$-space which consists of all the lattice-points satisfying the inequalities:

$$s_{r-1}+2 \leqq s_r \leqq w+2r \qquad (r=1,\ldots,u;\ s_0=q).$$

According to custom, an empty sum and an empty product should be understood to represent zero and unity, respectively. On the other hand, 0-dimensional range must be understood to consist of a single improper point; for instance,

$$\sum_{B_{q,w}^{(\nu)}} \prod_{r=1}^{0} 2^{\nu_{s_r}}=1.$$

3. Mother-descendant combination

We now consider a mother-descendant combination, of which the reduced probability is given by a defining equation

$$\kappa_{(1\nu;0)_t|n}(\alpha\beta;\ \xi\eta)=\sum \kappa_{(1\nu;0)_{t-1}|1\nu_t}(\alpha\beta;\ ab,\ cd)\varepsilon_n(ab,\ cd;\ \xi\eta).$$

Restricting ourselves to the case $\nu_r>1$ for $1\leqq r\leqq t$, we distinguish two systems according to $n=1$ and $n>1$. It can then be shown that *the desired formula is givin by*

$$\kappa_{(1\nu;0)_t|1}(\alpha\beta;\ \xi\eta)=\bar{A}_{\xi\eta}+[\bar{A}_2Q(\mathbf{0};\ \mathbf{1})]_{t+1}Q(\alpha\beta;\ \xi\eta)$$
$$+[\bar{A}_2R(\mathbf{1})]_{t+1}R(\xi\eta)+[\bar{A}_2T(\mathbf{0};\mathbf{1})]_{t+1}T(\alpha\beta;\xi\eta)+[\bar{A}_2S(\mathbf{0};\ \mathbf{1})]_{t+1}S(\alpha\beta;\xi\eta),$$
$$\kappa_{(1\nu;0)_t|n}(\alpha\beta;\ \xi\eta)=\bar{A}_{\xi\eta}+[\bar{A}_2Q(\mathbf{0};\ \mathbf{1})]_{t+1}\cdot 2^{-n+1}Q(\alpha\beta;\ \xi\eta)$$
$$=\bar{A}_{\xi\eta}+\Lambda_t 2^{-n+1}Q(\alpha\beta;\ \xi\eta) \qquad \text{for } n>1.$$

4. Descendants combination

A descendants combination is obtained, in general, by eliminating mother's type in the corresponding mother-descendants combination. It can be shown that *the probability of the descendants combination* $(\xi_1\eta_1,\ \xi_2\eta_2)_{(1\nu;0)_t|1\nu}$ *with* $\nu_r>1$ *for* $1\leqq r\leqq t+1$ *is represented by*

$$\sigma_{(1\nu;0)_t|1\nu}(\xi_1\eta_1,\ \xi_2\eta_2)=\bar{A}_{\xi_1\eta_1}\bar{A}_{\xi_2\eta_2}$$
$$+\{[\bar{A}_1Q(\mathbf{1};\ \mathbf{2})]_{t+1}+2[T(\mathbf{0};\ \mathbf{1},\ \mathbf{2})]_{t+1}\}\bar{A}_{\xi_1\eta_1}Q(\xi_1\eta_1;\ \xi_2\eta_2)$$
$$+\{[A_2T(\mathbf{0};\ \mathbf{1})]_{t+1}+[\bar{A}_2R(\mathbf{1})]_{t+1}\}\bar{A}_{\xi_2\eta_2}R(\xi_1\eta_1)$$
$$+\{[\mathfrak{R}^{\backprime}(\mathbf{1},\ \mathbf{2})]_{t+1}+2[\mathfrak{T}^{\backprime}(\mathbf{0};\ \mathbf{1},\ \mathbf{2})]_{t+1}\}\mathfrak{R}^{\backprime}(\xi_1\eta_1,\ \xi_2\eta_2).$$

5. Distribution of genotypes in a generation of descendant

By eliminating, for instance, mother's type from the probability of a mother-descendant combination, we get the distribution of genotypes in a corresponding descendant's generation. In case under consideration, *we get the formula*

$$\bar{A}_{(1\nu;0)_t|1}(\xi\eta)=\bar{A}_{\xi\eta}+\{[\bar{A}_2T(\mathbf{0};\ \mathbf{1})]_{t+1}+[\bar{A}_2R(\mathbf{1})]_{t+1}\}R(\xi\eta),$$
$$\bar{A}_{(1\nu;0)_t|n}(\xi\eta)=\bar{A}_{\xi\eta} \qquad \text{for } n>1.$$

A *deviation* of the same nature as frequently noticed appears here again in the generation immediate after the last consanguineous marriage.

6. Asymptotic behaviors of the probabilities

Asymptotic behaviors of every probability derived in the present chapter as anyone among the generation-numbers $\{\nu_r\}_{r=1}^{t+1}$ tends to infinity can be deduced from respective expression. For instance, we obtain the following limit equations

$$\lim_{\nu\to\infty} \kappa_{(1\nu;0)_t|1\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=\bar{A}_{\xi_2\eta_2}\kappa_{(1\nu;0)_t|1}(\alpha\beta;\ \xi_1\eta_1),$$

$$\lim_{\nu_w\to\infty} \kappa_{(1\nu;0)_t|1\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)=\kappa_{(1\nu;0)_{w-2}|1\nu_{w-1};1|(1\nu';0)_{t-w}|1\nu}(\alpha\beta;\ \xi_1\eta_1,\ \xi_2\eta_2)$$

where $\nu'_r\equiv\nu_{w+r}$ for $1\leqq r\leqq t-w$. Similar limit-relations can be readily written down for other combinations.

Asymptotic behaviors as $t\to\infty$ are, however, rather complicated in dealing. They depend on a sequence of generation-numbers $\{\nu_r\}_{r=1}^{\infty}$. We get, for instance,

$$\bar{A}_{ii}+\underline{\chi}R(ii)\leqq \varliminf_{t\to\infty} \bar{A}_{(1\nu;0)_t|1\nu}(ii)\leqq \varlimsup_{t\to\infty} \bar{A}_{(1\nu;0)_t|1\nu}(ii)\leqq \bar{A}_{ii}+\bar{\chi}R(ii),$$

$$\bar{A}_{ij}+\bar{\chi}R(ij)\leqq \varliminf_{t\to\infty} \bar{A}_{(1\nu;0)_t|1\nu}(ij)\leqq \varlimsup_{t\to\infty} \bar{A}_{(1\nu;0)_t|1\nu}(ij)\leqq \bar{A}_{ij}+\underline{\chi}R(ij),$$

where we put

$$\underline{\chi}=\varliminf_{t\to\infty}\ \{[\bar{A}_2T(\mathbf{0};\ \mathbf{1})]_{t+1}+[\bar{A}_2R(\mathbf{1})]_{t+1}\},$$

$$\bar{\chi}=\varlimsup_{t\to\infty}\ \{[\bar{A}_2T(\mathbf{0};\ \mathbf{1})]_{t+1}+[\bar{A}_2R(\mathbf{1})]_{t+1}\}$$

for which, in view of our assumption $\nu_r\geqq 2$ for any r, there hold the estimating inequalities

$$4/(2^{\bar{\nu}+2}-5)\leqq\underline{\chi}\leqq\bar{\chi}\leqq 4/(2^{\underline{\nu}+2}-5)$$

with

$$\bar{\nu}\equiv\varlimsup_{r\to\infty}\nu_r,\qquad \underline{\nu}\equiv\varliminf_{r\to\infty}\nu_r.$$