25. On the Convergence of Some Gap Series

By Noboru MATSUYAMA and Shigeru TAKAHASHI
Mathematical Institute, Kanazawa University, Japan
(Comm. by Z. SUETUNA, M.J.A., March 12, 1955)

§ 1. Let $f(x)$, $-\infty<x<+\infty$, be a function satisfying the following conditions:

(1.1) $$f(x+1)=f(x),$$

and

(1.2) $$\int_0^1 f(x)dx=0, \qquad \int_0^1 f^2(x)dx=1.$$

Further, let us put

(1.3) $$\omega(n)=\left(\int_0^1 \left|f(x)-s_n(x)\right|^2 dx\right)^{1/2}$$

where $s_n(x)$ denotes the n-th partial sum of the Fourier series of $f(x)$.

The following theorems were proved for the sequence $\{n_k\}$ of integers which has the Hadamard gap.

Theorem of M. Kac, R. Salem, and A. Zygmund [1]. *If*

(1.4) $$\omega(n)=O(1/(\log n)^{\alpha}), \qquad \alpha>1 \qquad (n\to+\infty)$$

and

(1.5) $$\sum c_n^2(\log n)^2<\infty,$$

then the series

(1.6) $$\sum c_k f(n_k x)$$

converges almost everywhere.

Theorem of S. Izumi [2]. *If*

(1.7) $$\omega(n)=O(1/n^{\alpha}), \qquad \alpha>0 \qquad (n\to+\infty)$$

and

(1.8) $$\sum c_n^2(\log_2 n)^2<+\infty,$$

then (1.6) converges almost everywhere.

The purpose of this paper is to generalize above results. Following G. Alexits [3], we shall say that a sequence $\{a_n\}$ is $\lambda(n)$-lacunary if

(1.9) $$[\text{the number of } n\text{'s such that } a_n\neq 0 \text{ for } 2^k\leq n<2^{k+1}]=O(\lambda(k)) \qquad (k\to+\infty),$$

where $\{\lambda(n)\}(n=0,1,2,\dots)$ is a non-decreasing sequence of positive numbers.

In the following, we shall assume that the sequence $\{a_n\}$ is $\lambda(n)$-lacunary and treat the convergence problem of the series

(1.10) $$\sum_{k=1}^{\infty} a_k f(kx).$$

In §§ 3–5, we prove the following theorems.

Theorem 1. *If (1.4) is satisfied and*

$$\sum_{n=1}^{\infty}\lambda(n)\sum_{k=2^n}^{2^{n+1}-1}a_k^2<+\infty, \tag{1.11}$$

then the almost everywhere (C, 1) summability of (1.10) implies the almost everywhere convergence of (1.10).

Theorem 2. *If (1.4) is satisfied and*

$$\sum_{n=1}^{\infty}\lambda(n)(\log n)^2\sum_{k=2^n}^{2^{n+1}-1}a_k^2<+\infty, \tag{1.12}$$

then (1.10) converges almost everywhere.

Theorem 3. *If (1.7) is satisfied and*

$$\sum_{n=1}^{\infty}\lambda(n)\,(\log_2 n)^2\sum_{k=2^n}^{2^{n+1}-1}a_k^2<+\infty, \tag{1.13}$$

then (1.10) converges almost everywhere.

§ 2. **Lemma 1.** *If (1.4) is satisfied, then for any i and j such that $2^n\leqq i<2^{n+1}$, $2^{n+k}\leqq j<2^{n+k+1}$, we have*

$$\left|\int_0^1 f(it)f(jt)dt\right|=O\left(\frac{1}{k^\alpha}\right)\qquad (k\to+\infty). \tag{2.1}$$

Proof. This is Lemma 1, [2].

For the convenience, we put

$$T_k(x)=\sum_{i=2^k}^{2^{k+1}-1}a_i f(ix),\qquad b_k^2=\sum_{i=2^k}^{2^{k+1}-1}a_i^2. \tag{2.2}$$

Lemma 2. *If (1.4) is satisfied, then*

$$\int_0^1\left(\sum_{k=0}^{n}T_k(x)\right)^2dx\leqq A\sum_{k=0}^{n}\lambda(k)b_k^2, \tag{2.3}$$

and

$$\int_0^1\operatorname*{Max}_{0\leqq j\leqq n}\left(\sum_{k=0}^{j}T_k(x)\right)^2dx\leqq A(\log n)^2\sum_{k=0}^{n}\lambda(k)b_k^2. \tag{2.4}$$

Proof. We have

$$\int_0^1\left(\sum_{k=0}^{n}T(x)\right)^2dx=\sum_{k=0}^{n}\int_0^1 T_k^2(x)dx+2\sum_{0\leqq k<k'\leqq n}\int_0^1 T_k(x)T_{k'}(x)dx\equiv I_1+2I_2.$$

From (2.2), we have $|I_1|\leqq\sum_{k=0}^{n}\lambda(k)b_k^2$, and, by (2.1)

$$\left|\int_0^1 T_k(x)T_{k'}(x)dx\right|=\left|\sum_{i=2^k}^{2^{k+1}-1}\sum_{j=2^{k'}}^{2^{k'+1}-1}a_i a_j\int_0^1 f(ix)f(jx)dx\right| \tag{2.5}$$

$$\leqq\frac{A}{(k'-k)^\alpha}\sum_{i=2^k}^{2^{k+1}-1}|a_i|\sum_{j=2^{k'}}^{k'+1-1}|a_j|\leqq\frac{A}{(k'-k)^\alpha}b_k b_k'(\lambda(k)\lambda(k'))^{1/2}.$$

Hence, we have

$$|I_2|\leqq A\sum_{0\leqq k\leqq k'\leqq n}\frac{1}{(k'-k)^\alpha}b_k b_{k'}(\lambda(k)\lambda(k'))^{1/2}\leqq A\sum_{k=0}^{n}\lambda(k)b_k^2.$$

Thus (2.3) is proved.

(2.4) is an analogue of the Menchoff's lemma [4]. Proof runs on the similar lines as this. The difference lies in the point that the Bessell's inequality is replaced by (2.3).

§3. **Proof of Theorem 1.** Let us put $\tau_n(x)=S_{2^n-1}(x)-\sigma_{2^n-1}(x)$ where $S_n(x)$ and $\sigma_n(x)$ denote the n-th partial sum and $(C,1)$ mean of the series (1.10) respectively. By (2.3) and (1.11) we have

$$\sum_{n=1}^{\infty}\int_0^1 \tau_n^2(x)dx=\sum_{n=1}^{\infty}\frac{1}{(2^n-1)^2}\int_0^1\Big[\sum_{k=1}^{2^n-1}(k-1)a_kf(kx)\Big]^2dx$$

$$\leqq A\sum_{n=1}^{\infty}\frac{1}{2^{2n}}\sum_{k=1}^{n-1}\lambda(k)\sum_{m=2^k}^{2^{k+1}-1}a_m^2(m-1)^2\leqq A\sum_{k=1}^{\infty}\lambda(k)\sum_{m=2^k}^{2^{k+1}-1}a_m^2(m-1)^2\sum_{n=k+1}^{\infty}\frac{1}{2^{2n}}$$

$$\leqq A\sum_{k=1}^{\infty}\lambda(k)b_k^2<+\infty.$$

This shows that the almost everywhere convergence of $\sigma_{2^n-1}(x)$ implies that of $S_{2^n-1}(x)$. We have

$$\int_0^1 \operatorname*{Max}_{2^n\leqq m<2^{n+1}}\Big(\sum_{k=2^n}^{m}a_kf(kx)\Big)^2dx\leqq\int_0^1\Big(\sum_{k=2^n}^{2^{n+1}-1}|a_kf(kx)|\Big)^2dx\leqq\lambda(n)b_n^2,$$

and then, by (1.11),

$$\sum_{n=1}^{\infty}\int_0^1 \operatorname*{Max}_{2^n\leqq m<2^{n+1}}\Big(\sum_{k=2^n}^{m}a_kf(kx)\Big)^2dx<+\infty.$$

Thus the theorem is proved.

§4. **Proof of Theorem 2.** From the preceding proof, it is sufficient to prove that the sequence

$$\sum_{k=0}^{n}T_k(x)=S_{2^{n+1}-1}(x)$$

converges almost everywhere.

By (1.12) and (2.3), there exists a function $F(x)$ of the L_2-class such that for any $\varepsilon<0$

$$\Big(\int_0^1\Big|F(x)-\sum_{k=0}^{m}T_k(x)\Big|^2dx\Big)^{1/2}<\varepsilon\qquad m>M=M(\varepsilon).$$

Thus we have

$$\Big(\int_0^1\Big(F(x)-\sum_{k=0}^{n}T_k(x)\Big)^2dx\Big)^{1/2}$$

$$\leqq\Big(\int_0^1\Big(F(x)-\sum_{k=0}^{m}T_k(x)\Big)^2dx\Big)^{1/2}+\Big(\int_0^1\Big(\sum_{k=n+1}^{m}T_k(x)\Big)^2dx\Big)^{1/2}$$

$$\leqq A\Big(\sum_{k=n+1}^{m}\lambda(k)b_k^2\Big)^{1/2}+\varepsilon\leqq A\Big(\sum_{n+1}^{\infty}\lambda(k)b_k^2\Big)^{1/2}=r_n^{1/2}.$$

By (1.12), it follows that

$$\sum_{n=1}^{\infty}r_{2^n}=\sum_{k=1}^{\infty}k(r_{2^k}-r_{2^{k+1}})+\lim_{n\to\infty}nr_{2^{n+1}}$$

$$=\sum_{k=1}^{\infty}k\lambda(2^{k+1})b_{2^{k+1}}^2\leqq\sum_{k=1}^{\infty}(\log k)\,\lambda(k)b_k^2<+\infty.$$

This shows that the sequence $\sum_{k=0}^{2^n} T_k(x)$ converges almost everywhere. Now, by (1.12) and (2.4), we have

$$\sum_{n=1}^{\infty}\int_0^1 \underset{2^n\leqq m<2^{n+1}}{\mathrm{Max}}\left(\sum_{k=2^n}^{m} T_k(x)\right)^2 dx \leqq A\sum_{n=1}^{\infty}(\log 2^n)^2 \sum_{k=2^n}^{2^{n+1}-1}\lambda(k)b_k^2$$

$$\leqq A\sum_{n=1}^{\infty}\sum_{k=2^n}^{2^{n+1}-1}(\log k)^2\lambda(k)b_k^2<+\infty.$$

This shows that the series $\sum_{k=0}^{\infty} T_k(x)$ converges almost everywhere.

§ 5. **Proof of Theorem 3.** Let us put

$$(5.1)\qquad U_k(x)=\sum_{i=2^k}^{2^{k+1}-1} a_i s_{\mu_k}(ix),\quad V_k(x)=T_k(x)-U_k(x)$$

where $\mu_k=[k^{1/2\alpha}(\log k)^{\beta/2\alpha}]$, $\beta>1$, for $k=2,3,\dots$.
Then by (1.1) and (1.7) we have

$$\int_0^1\sum_{k=2}^{\infty}\left|V_k(x)\right|dx\leqq\int_0^1\sum_{k=2}^{\infty}\left|f(x)-s_{\mu_k}(x)\right|\cdot\left|\sum_{i=2^k}^{2^{k+1}-1}a_i\right|dx$$

$$\leqq\int_0^1\sum_{k=2}^{\infty}\left|f(x)-s_{\mu_k}(x)\right|\cdot\left|\lambda(k)b_k^2\right|^{1/2}dx$$

$$\leqq\left(\int_0^1\left(\sum_{k=2}^{\infty}\left|f(x)-s_{\mu_k}(x)\right|^2\right)^{1/2}dx\right)\left(\sum_{k=2}^{\infty}\lambda(k)b_k^2\right)^{1/2}$$

$$\leqq A\left(\sum_{k=2}^{\infty}\int_0^1\left|f(x)-s_{\mu_k}(x)\right|^2dx\right)^{1/2}\leqq A\left(\sum_{k=2}^{\infty}\frac{1}{k(\log k)^{\beta}}\right)^{1/2}<+\infty.$$

Hence the series $\sum|V_k(x)|$ converges almost everywhere and in the L_2-mean.

On the other hand, by (1.13) and (2.3), the series $\sum T_k(x)$ converges in the L_2-mean, and hence the series $\sum U_k(x)$ converges in the L_2-mean, and then is the Fourier series of a function of the L_2-class. The m_k-th term in the series $\sum U_k(x)$ is the trigonometrical polynomial with the first term

$$a_{2^{m_k}}[A_1\cos(2\pi 2^{m_k}x)+B_1\sin(2\pi 2^{m_k}x)],$$

and with the last term

$$a_{2^{m_k+1}-1}[A_{\mu_{m_k}}\cos\{2\pi(2^{m_k+1}-1)\mu_{m_k}x\}+B_{\mu_{m_k}}\sin\{2\pi(2^{m_k+1}-1)\mu_{m_k}x\}],$$

where A_n and B_n are the n-th Fourier coefficients of $f(x)$.
Let us now put

$$\sum U_k(x)=\sum c_kU_k(x)+\sum c_k'U_k(x)$$

where $c_k'=1-c_k$ and

$$c_k=\begin{cases}1 & \text{for } m_{2v-1}<k\leqq m_{2v}\qquad (v=1,2,\dots),\\ 0 & \text{otherwise.}\end{cases}$$

If we take

$$m_k=[k(\log k)^{\gamma}],\qquad \gamma>1,$$

then

$$B(\log k)^{\gamma}>m_{k+1}-m_k>B'(\log k)^{\gamma},$$

where B and B' are positive constants independent of k. From the definitions of m_k and μ_k we have, for $k \geqq k_0$,

$$(2^{m_{2k}+1}-1)\mu_{m_{2k}} < 2^{m_{2k}+1},$$

and

$$2\cdot(2^{m_{2k}+1}-1)\mu_{m_{2k}} < (2^{m_{2k+2}+1}-1)\mu_{m_{2k+2}}.$$

Hence by Kolmogoroff's theorem [5] the series $\sum c_k U_k$ converges almost everywhere, and the same holds for $\sum c'_k U_k$, and hence the sequence $\sum_{i=0}^{m_k} T_i(x)$ converges almost everywhere.

On the other hand

$$\int_0^1 \operatorname{Max}_{m_k<n\leqq m_{k+1}} \left| \sum_{i=m_k}^{n} T_i(x) \right|^2 dx \leqq A \sum_k (\log (m_{k+1}-m_k))^2 \sum_{i=m_k+1}^{m_k+1} \lambda(i) b_i^2$$

$$\leqq A \sum_k (\log_2 k)^2 \sum_{i=m_k+1}^{m_k+1} \lambda(i) b_i^2 \leqq A \sum_k \sum_{i=m_k+1}^{m_{k+1}} (\log_2 i)^2 \lambda(i) b_i^2,$$

which is finite. Thus, by the familiar way, we get the almost everywhere convergence of the series $\sum_{i=0}^{\infty} T_i(x)$. Thus the theorem is proved.

References

[1] M. Kac, R. Salem, and A. Zygmund: A gap theorem, Trans. Amer. Math. Soc., **67**, 235–243 (1949).

[2] S. Izumi: Notes on Fourier analysis (XLI), Tôhoku Math. Jour., **3**, 89–103 (1951).

[3] G. Alexits: Sur la convergence des series lacunaires, Acta Sci. Math., **11**, 251–253 (1948); **13**, 14–17 (1949).

[4] S. Kacgmarz und H. Steinhaus: Theorie der Orthogonalreihen.

[5] A. Zygmund: Trigonometrical series.