48. Some Trigonometrical Series. XII

By Shin-ichi IZUMI
Mathematical Institute, Tokyo Metropolitan University, Tokyo
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1. A. Zygmund [1] has proved the following theorems.

Theorem 1. *Let $a(x)$ be a positive, decreasing and convex function in the interval $(0, \infty)$ such that*

(1) $$a(x) \downarrow 0, \quad xa(x) \uparrow \qquad \text{as } x \uparrow \infty.$$

Let $a_n = a(n)$ and

(2) $$\bar{f}(x) = \sum_{n=1}^{\infty} a_n \sin nx,$$

then we have

(3) $$\bar{f}(x) \sim x^{-1}a(x^{-1}) \qquad \text{as } x \to 0.$$

Theorem 2. *Let $a(x)$ be a positive, decreasing and convex function in the interval $(0, \infty)$, tending to zero as $x \to \infty$. Let $a_n = a(n)$ and suppose that*

(4) $$n\Delta a_n \downarrow, \ \sum a_n = \infty.$$

If we put

(5) $$f(x) = \sum_{n=1}^{\infty} a_n \cos nx,$$

then we have

(6) $$f(x) \sim \int_0^{1/x} t\,|a'(t)|\,dt \qquad \text{as } x \downarrow 0.$$

Omitting the second condition of $a(x)$ in (1), we prove the following

Theorem 3. *Let $a(x)$ be a positive, decreasing and convex function in the interval $(0, \infty)$, tending to zero as $x \to \infty$. Let $a_n = a(n)$ and define $\bar{f}(x)$ by (2), then*

(7) $$\bar{f}(x) \sim x \int_0^{1/x} ta(t)dt \qquad \text{as } x \to 0,$$

when $\bar{f}(x)$ is not bounded or the right side is ultimately positive.

If the second condition of (1) is satisfied, then we can easily see that (7) becomes (3).

In Theorem 2 we can replace the first condition of (4) by $\Delta^3 a_n \leqq 0$, that is,

Theorem 4. *Let $a(x)$ be a positive, decreasing and convex function in the interval $(0, \infty)$, tending to zero as $x \to \infty$ and let $-a'(t)$ be convex. Let $a_n = a(n)$ and suppose that $\sum a_n = \infty$. Then*

$$f(x) \sim \int_0^{1/x} t\,|a'(t)|\,dt \qquad \text{as } x \to 0.$$

Our proof of these theorems is very simple, except that the following lemma is used [2] (cf. [3]):

Lemma. *If* $a_n \downarrow 0$, *then*

$$\bar{f}(x)=\int_0^\infty a(t)\sin xt\,dt+\bar{g}(x),$$

$$f(x)=\int_0^\infty a(t)\cos xt\,dt+g(x),$$

where $\bar{g}(x)$ *and* $g(x)$ *are bounded.*

2. We shall prove Theorem 3. By Lemma, it is sufficient to prove that

$$\bar{f}_1(x)=\int_0^\infty a(t)\sin xt\,dt$$

satisfies the relation (8). We have

$$\bar{f}_1(x)=\int_0^{\pi/2x} b_x(t)\sin xt\,dt,$$

where

$$b_x(t)=a(t)+\sum_{k=1}^\infty(-1)^{k+1}\Big[a\Big(\frac{k\pi}{x}-t\Big)-a\Big(\frac{k\pi}{x}+t\Big)\Big]$$

$$=\sum_{k=0}^\infty(-1)^k\Big[a\Big(\frac{k\pi}{x}+t\Big)+a\Big(\frac{(k+1)\pi}{x}-t\Big)\Big].$$

By the monotonity of $a(t)$, we get

$$b_x(t)\leqq a(t)+a\Big(\frac{\pi}{x}-t\Big)$$

and by the convexity of $a(t)$, we get

$$b_x(t)\geqq a(t).$$

Hence

$$\bar{f}_1(x)\geqq\frac{2}{\pi}x\int_0^{\pi/2x}ta(t)dt\geqq Ax\int_0^{1/x}ta(t)dt \tag{8}$$

and

$$\bar{f}_1(x)\leqq x\int_0^{\pi/2x}ta(t)dt+x\int_{\pi/2x}^{\pi/x}\Big(\frac{\pi}{x}-t\Big)a(t)dt$$

$$\leqq Ax\int_0^{\pi/2x}ta(t)dt\leqq Ax\int_0^{1/x}ta(t)dt. \tag{9}$$

Thus we get (7).

We shall now prove the following

Theorem 5. *Let* $a(x)$ *be a positive decreasing sequence such that there is a positive constant* $c<1$ *such that*

$$a(t)>ca(3t)\qquad(t>0). \tag{10}$$

Let $a_n=a(n)$ *and define* $\bar{f}(x)$ *by (2), then (7) holds when* $\bar{f}(x)$ *is unbounded or the right of (7) is ultimately positive.*

In the proof of Theorem 3, we did not use the convexity of (a_n) to prove (9). We shall prove (8) by the condition (10). By the monotonity of $a(t)$,

$$\bar{f}_1(x) \geqq \int_0^{\pi/2x} a(t) \sin xt + \int_{3\pi/2x}^{2\pi/x} a(t) \sin xt\, dt$$
$$= \int_0^{\pi/2x} \left[a(t) - a\left(\frac{2\pi}{x} - t\right) \right] \sin xt\, dt$$
$$\geqq Ax \int_0^{\pi/2x} ta(t) dt.$$

Thus we get (8), and hence the theorem is proved.

3. Let us now prove Theorem 4. Let

$$f_1(x) = \int_0^\infty a(t) \cos xt\, dt$$
$$= -\frac{1}{x} \int_0^\infty a'(t) \sin xt\, dt = -\frac{1}{x} \int_0^{\pi/2x} b_x'(t) \sin xt\, dt,$$

where $b_x'(t)$ denotes the term-wise differentiated series of $b_x(t)$ by t. Hence, from the proof of Theorem 3, we get the required result.

Finally we shall prove the following

Theorem 6. *Let $a(x)$ be a positive, decreasing and convex function in the interval $(0, \infty)$, tending to zero as $x \to \infty$. Let $a_n = a(n)$ and $\sum a_n = \infty$. Then*

$$f(x) \leqq A \int_0^{1/x} t\, |a'(t)|\, dt \qquad as\ x \to 0.$$

For, we write

$$f_1(x) = \int_0^\infty a(t) \cos xt\, dt = \int_0^{\pi/2x} c_x(t) \cos xt\, dt,$$

where

$$c_x(t) = \sum_{k=0}^\infty (-1)^k \left[a\left(\frac{k\pi}{x} + t\right) - a\left(\frac{(k+1)\pi}{x} - t\right) \right].$$

By the convexity of $a(t)$,

$$c_x(t) \leqq a(t) - a(\pi/x - t),$$

and then

$$f_1(x) \leqq \int_0^{\pi/2x} \cos xt [a(t) - a(\pi/x - t)] dt$$
$$= -\int_0^{\pi/2x} \cos xt\, dt \int_t^{\pi/x - t} a'(u) du$$
$$= -\int_0^{\pi/2x} a'(u) du \int_0^u \cos xt\, dt - \int_{\pi/2x}^{\pi/x} a'(u) du \int_0^{\pi/x - u} \cos xt\, dt$$
$$\leqq -\frac{A}{x} \int_0^{\pi/2x} a'(u) \sin xu\, du \leqq -A \int_0^{\pi/2x} a'(u) u\, du.$$

Thus we get the required inequality.

References

[1] A. Zygmund: Trigonometrical series, Warszawa, 114–116 (1936).
[2] R. Salem: Comptes Rendus, **207** (1939).
[3] S. Izumi and M. Satô: Tôhoku Math. Journ., **6** (1954).