64. Representation Theorems of Operator Algebra and Their Applications

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Recently M. Tomita has obtained remarkable results concerning the irreducible decomposition theory of C^*-algebra with identity operator (cf. [6][1]). (C^*-algebra is a uniformly closed self-adjoint operator algebra on a Hilbert space.) In this present paper we shall first extend his results to the case of arbitrary C^*-algebra without identity operator. It seems to us that these results are the best generalization of tasks of M. Nakamura-Y. Misonou [3], R. Godement [1], and H. Umegaki [8]. Secondly, applying this decomposition theorem to non-separable locally compact groups, we shall have an irreducible decomposition of unitary representation. It contains a generalization of some results of R. Godement [2]. Throughout this paper, we shall use notations, terminologies and results of [6].

From M. Tomita's letter we know that he obtained results analogous to ours. His results not yet known to us will be published in the near future. Our special thanks are given to M. Tomita, whose kindness made this publication of our paper possible.

1. Decompositions of states and traces. Any state p given on C^*-algebra $\boldsymbol{A}$ in which identity operator I does not necessarily exist, there exists a normal representation $\{\boldsymbol{A}_p, L^2(p), \hat{p}\}$ of $\boldsymbol{A}$ with the following properties:

(1) $\boldsymbol{A}_p$ is a representative algebra of $\boldsymbol{A}$ on a Hilbert space $L^2(p)$, i.e. there exists a continuous $*$-algebraic homomorphism $A \to A_p$ of $\boldsymbol{A}$ onto a dense sub-algebra of $\boldsymbol{A}_p$;

(2) a set $(A_p\hat{p} : A \in \boldsymbol{A})$ is everywhere dense in $L^2(p)$;

(3) $p(A)=(A_p\hat{p}, \hat{p})$ for every $A \in \boldsymbol{A}$, where (,) is the inner product of $L^2(p)$.

Conversely, a normal representation $\{\boldsymbol{A}_p, L^2(p), \hat{p}\}$ of $\boldsymbol{A}$ given, p defined by $p(A)=(A_p\hat{p}, \hat{p})$ is a state on $\boldsymbol{A}$. This correspondence between states and normal representations is unique within equivalence (I. E. Segal's well-known theorem: Theorem 1 of [5]).

Now let $\bar{\boldsymbol{A}}$ be C^*-algebra obtained by adjoining identity operator I to C^*-algebra $\boldsymbol{A}$. Let p be a state on $\boldsymbol{A}$ and let $\{\boldsymbol{A}_p, L^2(p), \hat{p}\}$ be the corresponding normal representation. And let $\bar{\boldsymbol{A}}_p$ denote C^*-

1) Numbers in brackets refer to the references at the end of this paper.

algebra obtained by supplement of identity operator I_p on $L^2(p)$ to $\boldsymbol{A}_p$. Then $\{\bar{\boldsymbol{A}}_p, L^2(p), \hat{p}\}$ is plainly a normal representation of $\bar{\boldsymbol{A}}$.

The diagonal algebra $\boldsymbol{E}$ of $\bar{\boldsymbol{A}}_p$, i.e. a commutative algebra $\boldsymbol{E}=(\boldsymbol{E}\cup\bar{\boldsymbol{A}}_p)'$[2] on $L^2(p)$ is also a diagonal algebra of $\boldsymbol{A}_p$, because we have $(\boldsymbol{E}\cup\boldsymbol{A}_p)'=\boldsymbol{E}$. Let us put $\boldsymbol{R}=\boldsymbol{E}\cup\bar{\boldsymbol{A}}_p=\boldsymbol{E}\cup\boldsymbol{A}_p$. Then the linear functional t on $\boldsymbol{R}$ defined by $t(R)=(R\hat{p}, \hat{p})$ for every $R\in\boldsymbol{R}$ is a state on $\boldsymbol{R}$. Moreover $\{\boldsymbol{R}_t, L^2(t), \hat{t}\}$, where $L^2(t)=L^2(p)$, $\hat{t}=\hat{p}$, and $\boldsymbol{R}_t=\boldsymbol{R}$, is clearly the normal representation corresponding to the state t. Now the state t is reducible state (i.e. for every state $r\leqq t$, there exists at least one $0\leqq K\in\boldsymbol{E}$ (the center of $\boldsymbol{R}$) which satisfies $r(R)=t(KR)$ for every $R\in\boldsymbol{R}$), and since the contraction t_E of t to $\boldsymbol{E}$ is a state on the commutative algebra $\boldsymbol{E}$, it determines a non-negative Borel measure on the spectrum $\mathfrak{E}$ of $\boldsymbol{E}$. We denote the measure by the same letter t_E. Then the support of t_E coincides with $\mathfrak{E}$, and we shall obtain M. Tomita's spectral decomposition of the state t:

$$t=\int_{\mathfrak{E}}\lambda^t dt_E(\lambda),$$

where λ^t (for $\lambda\in\mathfrak{E}$) are the states on $\boldsymbol{R}$ called derivative states of t, and almost all λ^t in reference to the measure t_E are irreducible (i.e. in the terminology of I. E. Segal, pure) (cf. Lemmas 5.2, 5.3, Theorems 2 and 3 of [6]).

To make notations brief, we shall denote $L^2(\lambda^t)$, $\hat{\lambda}^t$, R_{λ^t} (for $R\in\boldsymbol{R}$), $\bar{A}_{p\lambda^t}$ (for $\bar{A}\in\bar{\boldsymbol{A}}$), $A_{p\lambda^t}$ (for $A\in\boldsymbol{A}$), $\boldsymbol{R}_{\lambda^t}$, the uniform closure of $(\bar{A}_{p\lambda^t}: \bar{A}\in\bar{\boldsymbol{A}})$ and the uniform closure of $(A_{p\lambda^t}: A\in\boldsymbol{A})$ respectively by $L^2(\lambda)$, $\hat{\lambda}$, R_λ, $\bar{A}_\lambda$, A_λ, $\boldsymbol{R}_\lambda$, $\bar{\boldsymbol{A}}_\lambda$, and $\boldsymbol{A}_\lambda$ for every $\lambda\in\mathfrak{E}$. Then $\{\boldsymbol{R}_\lambda, L^2(\lambda), \hat{\lambda}\}$ (for $\lambda\in\mathfrak{E}$) are normal representations of $\boldsymbol{R}$, and almost all $\{\boldsymbol{R}_\lambda, L^2(\lambda), \hat{\lambda}\}$ are irreducible. Secondly, for each $\lambda\in\mathfrak{E}$ λ^p defined by $\lambda^p(A)=\lambda^t(A_p)$ for every $A\in\boldsymbol{A}$ is a state on $\boldsymbol{A}$. Let $\{\boldsymbol{A}_{\lambda^p}, L^2(\lambda^p), \hat{\lambda^p}\}$ be the normal representation corresponding to the state λ^p, and let $\bar{\boldsymbol{A}}_{\lambda^p}$ be C^*-algebra obtained by adjoining I_{λ^p} to $\boldsymbol{A}_{\lambda^p}$. Then $\{\bar{\boldsymbol{A}}_{\lambda^p}, L^2(\lambda^p), \hat{\lambda^p}\}$ is a normal representation of $\bar{\boldsymbol{A}}$. On the other hand, putting $\bar{\lambda^p}(\bar{A})=\lambda^t(\bar{A}_p)$ for every $\bar{A}\in\bar{\boldsymbol{A}}$, $\bar{\lambda^p}$ is a state of $\bar{\boldsymbol{A}}$. Then $\{\bar{\boldsymbol{A}}_{\bar{\lambda^p}}, L^2(\bar{\lambda^p}), \hat{\bar{\lambda^p}}\}$, where $L^2(\bar{\lambda^p})=L^2(\lambda)$, $\hat{\bar{\lambda^p}}=\hat{\lambda}$, and $\bar{\boldsymbol{A}}_{\bar{\lambda^p}}=\bar{\boldsymbol{A}}_\lambda$, is also a normal representation of $\bar{\boldsymbol{A}}$, and we have $\bar{\boldsymbol{A}}_{\bar{\lambda^p}}=\boldsymbol{R}_\lambda$ (cf. Lemma 5.4 of [6]). But since we have $\bar{\lambda^p}(\bar{A})=(\bar{A}_{\bar{\lambda^p}}\hat{\bar{\lambda^p}}, \hat{\bar{\lambda^p}})=(\bar{A}_\lambda\hat{\lambda}, \hat{\lambda})=(\bar{A}_{\lambda^p}\hat{\lambda^p}, \hat{\lambda^p})$, two normal representations $\{\bar{\boldsymbol{A}}_{\lambda^p}, L^2(\lambda^p), \hat{\lambda^p}\}$ and $\{\bar{\boldsymbol{A}}_{\bar{\lambda^p}}, L^2(\bar{\lambda^p}), \hat{\bar{\lambda^p}}\}$ are equivalent. And since $\{\bar{\boldsymbol{A}}_{\lambda^p}, L^2(\lambda^p),$

2) Let $\boldsymbol{M}\cup\boldsymbol{N}$ denote the smallest C^*-algebra which contains two C^*-algebras $\boldsymbol{M}$ and $\boldsymbol{N}$, and $\boldsymbol{M}'$ is the commutator of $\boldsymbol{M}$.

$\hat{\lambda^p}\}$ is irreducible if and only if λ^t is irreducible, $\{\bar{A}_{\lambda^p}, L^2(\lambda^p), \hat{\lambda^p}\}$, therefore, $\{A_{\lambda^p}, L^2(\lambda^p), \hat{\lambda^p}\}$ is irreducible if and only if λ^t is irreducible. Then we have the following:

Lemma 1. Almost all states λ^p (for $\lambda \in \mathfrak{E}$) in reference to the measure t_E are irreducible.

And also we have in general the following lemma:

Lemma 2. If r is a center-reducible[3] *trace (i.e. center-reducible central state) on $\boldsymbol{A}$, then derivative states λ^r of r are traces.*

Proof. By Theorem 2 of [6], if r is center-reducible, then we have

$$r(KA)=\int K(\lambda)\lambda^r(A)dr_Z(\lambda) \quad \text{for every } A \in \boldsymbol{A},\ K \in \boldsymbol{Z} \text{ (the center).}$$

And if r is a trace, then we have $r(KAB)=r(KBA)$ for every $A, B \in \boldsymbol{A}$, $K \in \boldsymbol{Z}$. Therefore we have

$$\int K(\lambda)\lambda^r(AB)dr_Z(\lambda)=\int K(\lambda)\lambda^r(BA)dr_Z(\lambda) \quad \text{for every } A, B \in \boldsymbol{A},\ K \in \boldsymbol{Z},$$

that is, $\quad \lambda^r(AB)=\lambda^r(BA) \quad$ for every $A, B \in \boldsymbol{A}$. Q.E.D.

Then putting $\tau=t_E$, we obtain the following theorem:

THEOREM 1. *Let p be a state on $\boldsymbol{A}$ in which identity operator I does not necessarily exist, and let $\boldsymbol{E}$ be the diagonal algebra on $L^2(p)$. Then there exists a reducible Borel measure τ on the spectrum $\mathfrak{E}$ of $\boldsymbol{E}$, and for each $\lambda \in \mathfrak{E}$ there corresponds the state λ^p on $\boldsymbol{A}$ with the following properties:*

(1) *$\lambda^p(A)$ is continuous in $\mathfrak{E}$ for every fixed $A \in \boldsymbol{A}$;*

(2) *almost all λ^p in reference to the measure τ have $\|\lambda^p\|=1$;*[4]

(3) $(A_p K\hat{p}, \hat{p})=\int_{\mathfrak{E}} K(\lambda)\lambda^p(A)d\tau(\lambda)$ *for every $A \in \boldsymbol{A}$, $K \in \boldsymbol{E}$;*

(4) *almost all λ^p in reference to the measure τ are irreducible;*

(5) *if p is a trace on $\boldsymbol{A}$, then almost all λ^p are characters (i.e. irreducible traces).*

Let $\mathfrak{P}$ denote the set of all states r on $\boldsymbol{A}$ with $\|r\| \leqq 1$, and let $\mathfrak{N}$ denote the set of all irreducible states t on $\boldsymbol{A}$ with $\|t\|=1$. Then the correspondence $\lambda \to \lambda^p$ is a weakly continuous mapping of $\mathfrak{E}$ into $\mathfrak{P}$, and almost all λ^p belong to $\mathfrak{N}$. Therefore, the image $\mathfrak{D}$ of $\mathfrak{E}$ with respect to the mapping is contained in the weak closure $\bar{\mathfrak{N}}$ of $\mathfrak{N}$. Let ρ be the Borel measure induced by τ in $\mathfrak{D}$. We may regard ρ as a Borel measure on $\bar{\mathfrak{N}}$ whose support is $\mathfrak{D}$. And almost all states in $\bar{\mathfrak{N}}$ in reference to the measure ρ are irreducible. Secondly, let $\boldsymbol{M}$ denote *-algebra of all essentially bounded ρ-measur-

3) If r is the state on $\boldsymbol{A}$ whose contraction r_Z to the center $\boldsymbol{Z}$ of $\boldsymbol{A}$ is a reducible state, then r is called center-reducible.

4) For instance, cf. Proof of Theorem 1 of I. E. Segal: Decompositions of operator algebras. I, Mem. Amer. Math. Soc., No. 9 (1951).

able functions ξ on $\overline{\mathfrak{N}}$. Then we have

THEOREM 2. *Let p be a state on $\boldsymbol{A}$, and let $\boldsymbol{E}$ be the diagonal algebra on $L^2(p)$. Then there exists a Borel measure ρ on $\overline{\mathfrak{N}}$ with the following properties:*

(1) *$\overline{\mathfrak{N}}-\mathfrak{N}$ is ρ-measure 0;*

(2) *for every $\xi \in \boldsymbol{M}$ there exists $K_\xi \in \boldsymbol{E}$ which satisfies that*

$$(A_p K_\xi \hat{p}, \hat{p}) = \int \xi(\omega)\omega(A)d\rho(\omega) \qquad \text{for every } A \in \boldsymbol{A};$$

(3) *the correspondence $\xi \to K_\xi$ is $*$-algebraic isometric isomorphism of $\boldsymbol{M}$ onto $\boldsymbol{E}$;*

(4) *if p is a trace, then almost all ω are characters.*

The proof of this theorem will not be made here as its proof is similar to that of Theorem 3 which we shall show below.

2. Applications to topological groups. Let G be an arbitrary locally compact group, and let $L^1(G)$ be its L^1-group algebra. If φ is any continuous positive definite function on G, then by the well-known method we may obtain a Hilbert space $L^2(\varphi)$. Let $f \to f_\varphi$ denote a mapping of $L^1(G)$ to a dense sub-set of $L^2(\varphi)$. If $U_{f;\varphi}$ is the representation on $L^2(\varphi)$ of $L^1(G)$, then we have $U_{f;\varphi}g_\varphi = (fg)_\varphi$[5] for every $f, g \in L^1(G)$, and there exists the element $\hat{\varphi} \in L^2(\varphi)$ which satisfies:

$$(U_{f;\varphi}\hat{\varphi}, \hat{\varphi}) = \int_G f(a)\varphi(a)da \qquad \text{for every } f \in L^1(G).$$

Let $\boldsymbol{A}_\varphi$ be the uniform closure of the set $(U_{f;\varphi} : f \in L^1(G))$, then $\boldsymbol{A}_\varphi$ is clearly C^*-algebra. We may define a state φ on $\boldsymbol{A}_\varphi$ by the following way: $\varphi(A_\varphi) = (A_\varphi\hat{\varphi}, \hat{\varphi})$ for every $A_\varphi \in \boldsymbol{A}_\varphi$. Since $\{\boldsymbol{A}_\varphi, L^2(\varphi), \hat{\varphi}\}$ is the normal representation corresponding to φ, by virtue of Theorem 1, if we denote the diagonal algebra on $L^2(\varphi)$ by $\boldsymbol{E}$, then we obtain the decomposition of φ: $\varphi = \int_{\mathfrak{S}} \lambda^\varphi d\tau(\lambda)$, and moreover we have

$$(A_\varphi K\hat{\varphi}, \hat{\varphi}) = \int_{\mathfrak{S}} K(\lambda)\lambda^\varphi(A_\varphi)d\tau(\lambda) \qquad \text{for every } A_\varphi \in \boldsymbol{A}_\varphi,\ K \in \boldsymbol{E}.$$

On the other hand, since λ^φ (for each $\lambda \in \mathfrak{S}$) is a state on $\boldsymbol{A}_\varphi$, we may have a bounded positive linear functional φ_λ on $L^1(G)$ which is defined by $\varphi_\lambda(f) = \lambda^\varphi(U_{f;\varphi})$ for every $f \in L^1(G)$. Therefore, there exists a Hilbert space $L^2(\varphi_\lambda)$, and let $U_{a;\varphi_\lambda}$ be the unitary representation on $L^2(\varphi_\lambda)$ of G, then by the well-known theorem we may obtain the continuous p. d. function φ_λ on G which is defined by $\varphi_\lambda(a) = (U_{a;\varphi_\lambda}\hat{\varphi}_\lambda, \hat{\varphi}_\lambda)$ for $a \in G$. And since φ_λ is an elementary continuous p. d. function on G if λ^φ is an irreducible state, almost all φ_λ are elemen-

5) Of course, $fg(a) = \int_G f(b)g(b^{-1}a)db$, $f^*(a) = \rho(a)(f(a^{-1}))^c$.

tary continuous p. d. functions on G.

Let $\mathfrak{P}(G)$ denote the set of all p. d. functions ψ on G with $\|\psi\| \leqq 1$ in the dual space $L^\infty(G)$ of $L^1(G)$, and let $\mathfrak{R}(G)$ denote the set of all elementary p. d. functions χ with $\|\chi\|=1$. Then $\lambda \to \varphi_\lambda$ is a weakly continuous mapping of $\mathfrak{E}$ into $\mathfrak{P}(G)$ and almost all φ_λ belong to $\mathfrak{R}(G)$. Therefore, the image $\mathfrak{D}(G)$ of $\mathfrak{E}$ with respect to the mapping is contained in the weak closure $\overline{\mathfrak{R}(G)}$ of $\mathfrak{R}(G)$. Let π be the Borel measure induced by τ in $\mathfrak{D}(G)$. Then we may regard π as a Borel measure on $\overline{\mathfrak{R}(G)}$ whose support is $\mathfrak{D}(G)$. And almost all χ in $\overline{\mathfrak{R}(G)}$ in reference to π are elementary. Now let $\boldsymbol{M}(G)$ denote *-algebra of all essentially bounded π-measurable functions ξ on $\overline{\mathfrak{R}(G)}$, and for every $\xi \in \boldsymbol{M}(G)$ let ξ^φ denote the τ-measurable function on $\mathfrak{E}$ which is defined by the following way: $\xi^\varphi(\lambda)=\xi(\varphi_\lambda)$ for every $\lambda \in \mathfrak{E}$. Then we have

Lemma 3. If ξ belongs in $\boldsymbol{M}(G)$, then there exists $K_\xi \in \boldsymbol{E}$ which satisfies $K_\xi(\lambda)=\xi^\varphi(\lambda)$ a.e. and $(U_{f;\varphi}K_\xi\hat{\varphi}, \hat{\varphi})=\int_{\overline{\mathfrak{R}(G)}} \chi(f)\xi(\chi)d\pi(\chi)$.

Proof. The generality of the lemma will not be lost under the assumption that $0 \leqq \xi \leqq 1$. Let τ_ξ denote the state on $\boldsymbol{E}=C(\mathfrak{E})$ which is defined by $\tau_\xi(E)=\int_{\mathfrak{E}} \xi^\varphi(\lambda)E(\lambda)d\tau(\lambda)$. Then we have $\tau_\xi \leqq \tau$. Therefore, by the reducibility of τ there exists $0 \leqq K_\xi \in \boldsymbol{E}$ which satisfies $\tau_\xi(E)=\tau(K_\xi E)=\int_{\mathfrak{E}} K_\xi(\lambda)E(\lambda)d\tau(\lambda)$ for every $E \in \boldsymbol{E}$. It follows from these that $K_\xi(\lambda)=\xi^\varphi(\lambda)$ a.e. and

$$(U_{f;\varphi}K_\xi\hat{\varphi}, \hat{\varphi})=\int_{\mathfrak{E}} \lambda^\varphi(U_{f;\varphi})\xi^\varphi(\lambda)d\tau(\lambda)=\int_{\mathfrak{D}(G)} \chi(f)\xi(\chi)d\pi(\chi).$$

This concludes the lemma. Q.E.D.

*Lemma 4. The c rrespondence $\xi \to K_\xi$ is *-algebraic isometric isomorphism of $\boldsymbol{M}(G)$ onto $\boldsymbol{E}$.*

Proof. For every $\xi \in \boldsymbol{M}(G)$ we have

$$\|\xi\|=ess.\ max._\lambda |\xi^\varphi(\lambda)|=sup_\lambda |K_\xi(\lambda)|=\|K_\xi\|.$$

Then $\xi \to K_\xi$ is *-algebraic isometric isomorphism of $\boldsymbol{M}(G)$ onto a subalgebra $\boldsymbol{F}$ of $\boldsymbol{E}$. Now in order to prove that $\boldsymbol{F}=\boldsymbol{E}$, it is sufficient to show that for every $K \in \boldsymbol{E}$ with $0 \leqq K \leqq I$ there exists $\xi \in \boldsymbol{M}(G)$ having $K_\xi=K$. For such $K \in \boldsymbol{E}$ let ψ denote a positive linear functional on $C(\overline{\mathfrak{R}(G)})$ which is defined by $\psi(x)=\int_{\mathfrak{E}} x^\varphi(\lambda)K(\lambda)d\tau(\lambda)$ for $x \in C(\overline{\mathfrak{R}(G)})$. Since $I \geqq K \geqq 0$, we have $\pi \geqq \psi \geqq 0$. Then ψ is absolutely continuous with respect to the measure π. Therefore, there exists a bounded π-measurable function ξ on $\mathfrak{D}(G) \subset \overline{\mathfrak{R}(G)}$ which satisfies $\psi(x)=\int \xi(\chi)x(\chi)d\pi(\chi)$. We shall now show that $K=K_\xi$. Since $K_\xi(\lambda)=\xi^\varphi(\lambda)$ a.e., we have $\int x^\varphi(\lambda)K(\lambda)d\tau(\lambda)=\int x^\varphi(\lambda)K_\xi(\lambda)d\tau(\lambda)$, therefore, especially for every $U_{f;\varphi}\in \boldsymbol{A}_\varphi$ we have $\int \lambda^\varphi(U_{f;\varphi})K(\lambda)d\tau(\lambda)$ $=\int \lambda^\varphi(U_{f;\varphi})K_\xi(\lambda)d\tau(\lambda)$; and we thus have $(U_{f;\varphi}K\hat{\varphi}, \hat{\varphi})=(U_{f;\varphi}K_\xi\hat{\varphi}, \hat{\varphi})$. It follows from this that $(KU_{f;\varphi}\hat{\varphi}, U_{g;\varphi}\hat{\varphi})=(K_\xi U_{f;\varphi}\hat{\varphi}, U_{g;\varphi}\hat{\varphi})$ for every $f, g\in L^1(G)$, and the set $(U_{f;\varphi}\hat{\varphi}: f\in L^1(G))$ is everywhere dense in $L^2(\varphi)$, therefore, we have $K=K_\xi$. Hence we have $\boldsymbol{F}=\boldsymbol{E}$ as we wished to prove. Q.E.D.

The next theorem now follows at once from that if $\{f^\alpha\}\subset L^1(G)$ is a singularly directed system with respect to $a\in G$ (cf. [4]), then $[U_{f^\alpha;\varphi}]$ tends weakly to $U_{a;\varphi}$.

THEOREM 3. *Let G be a locally compact group, let $\mathfrak{N}(G)$ be the set of all elementary positive definite functions χ on G with $\|\chi\|=1$ and let $\overline{\mathfrak{N}(G)}$ denote the weak closure of $\mathfrak{N}(G)$. Given a continuous positive definite function φ on G and the diagonal algebra $\boldsymbol{E}$ on $L^2(\varphi)$. Then there exists a Borel measure π with the following properties:*

(1) $\overline{\mathfrak{N}(G)}-\mathfrak{N}(G)$ is π-measure 0;

*(2) let $\boldsymbol{M}(G)$ be *-algebra of all essentially bounded π-measurable functions ξ on $\overline{\mathfrak{N}(G)}$, then for every $\xi\in \boldsymbol{M}(G)$ there exists $K_\xi\in\boldsymbol{E}$ which satisfies*

$$(U_{a;\varphi}K_\xi\hat{\varphi}, \hat{\varphi})=\int \xi(\chi)\chi(a)d\pi(\chi)\quad \textit{for every } a\in G;$$

*(3) the correspondence $\xi\rightarrow K_\xi$ is *-algebraic isometric isomorphism of $\boldsymbol{M}(G)$ onto $\boldsymbol{E}$;*

(4) if φ is a continuous central positive definite function on G, then almost all χ in reference to the measure π are elementary continuous central positive definite functions.

What is written in this paper is a mere summary. Details will be discussed elsewhere when published in a more general form accompanied by other statements.

References

[1] R. Godement: Sur la théorie des caractères. I et II, C.R. Paris, **229**, 976–979, 1050–1051 (1949).

[2] R. Godement: Mémoire sur la théorie des caractères dans les groupes localement compacts unimodulaires, J. Math. pures et appl., **30**, 1–110 (1951).

[3] M. Nakamura and Y. Misonou: Centering of an operator algebra, Tohoku Math. J., **3**, 243–248 (1951).

[4] D. A. Raikov: Harmonic analysis on commutative group with the Haar measure and theory of characters, Travaux de l'Inst. Math. Stekloff, **14**, 5–86 (1945).

[5] I. E. Segal: Irreducible representations of operator algebra, Bull. Amer. Math. Soc., **48**, 73–88 (1947).

[6] M. Tomita: Representations of operator algebra, Math. J. Okayama Univ., **3**, 147–173 (1954).

[7] K. Tsuji: N^*-algebras and finite class groups, appear in Bull. Kyushu Inst. Tech. (M. & N.S.), **1**, 1–9 (1955).

[8] H. Umegaki: Decomposition theorems of operator algebra and their applications, Jap. J. Math., **22**, 27–50 (1952).