61. On the Structure of Semigroups Containing Minimal Left Ideals and Minimal Right Ideals

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(Comm. by Z. SUETUNA, M.J.A., May 13, 1955)

For the semigroup S containing minimal left ideals and minimal right ideals, we can define the kernel D of S as the product of any one minimal left ideal L and minimal right ideal R. It should be noted that D does not depend on the choice of L and R. The previous works in which the structure of the kernel is treated have already been given in [1], [2], and [3]. Recently the present author has shown the following:

Let D_{kl}, $k=1,2,\dots,p;l=1,2,\dots,q$, where p is the number of minimal right ideals and q one of minimal left ideals in the kernel D, be the groups which compose D and have no element in common, then every minimal left ideal L and every minimal right ideal R can be represented in the form

$$L_l=\sum_{k=1}^{p}D_{kl}$$
$$R_k=\sum_{l=1}^{q}D_{kl}, \tag{1}$$

and among these groups,

$$D_{jl}=D_{jm}D_{kl} \tag{2}$$

holds true [4].

The purpose of the present paper is to show that the semigroup S containing minimal left ideals and minimal right ideals has the similar structure as that of the kernel under the condition weaker than the cancellation law.

In the present paper we use certain of the ideas, notations, and results given in the previous paper [4] without explanation.

We have already seen that d is an element of a minimal left ideal L_l, then $d\,\varepsilon\,D_{kl}$ for a certain k, and $Sd=L_l$. Let S_{kj} be the set of elements $s\,\varepsilon\,S$ such that $sd\,\varepsilon\,D_{jl}$, then $S=\sum_{j=1}^{p}S_{kj}$ and $S_{kl}\cap S_{kj}=\phi$ since $D_{il}\cap D_{jl}=\phi$ for $i\neq j$. We shall call this decomposition of S the s-decomposition.

For R_j, the minimal right ideal including D_{jl}, $R_jD_{kl}=D_{jl}$ by (2), therefore,

$$S_{kj}\supset R_j \qquad \text{for all } k. \tag{3}$$

Let d' be any element of D_{kl}, then for $s_{kj}\,\varepsilon\,S_{kj}$, $s_{kj}d'\,\varepsilon\,s_{kj}(dE)\,\varepsilon\,D_{jl}E=D_{jl}$ where E is any one of groups composing L_l, and so we have $S_{kj}\cdot d' \subset D_{jl}$. Therefore we have $S_{kj} \subset S'_{kj}$, where S'_{kj} is the set of elements s of S such that $sd' \varepsilon D_{jl}$. Analogously $S'_{kj} \subset S_{kj}$, then we have $S_{kj} = S'_{kj}$. So the s-decompositions of S by element of D_{kl} are the same.

Next, for $d'' \varepsilon D_{km}$, $s_{kj}d'' \varepsilon s_{kj}(dE)d'' \varepsilon D_{jl}(Ed'') = D_{jl}D_{lm} = D_{jm}$ where E is any one of the groups which compose L_l, so that, $S_{kj}D_{km} \subset D_{jm}$ is obtained. On the other hand,

$$S_{kj}D_{km} \supset R_jD_{km} = \sum_{l=1}^{q} D_{jl}D_{km} = D_{jm}$$

by (1), (2), and (3), and we have

$$S_{kj}D_{km} = D_{jm} \qquad \text{for all } m. \tag{4}$$

It should be noted that generally $S_{ij} \neq S_{kj}$ for $i \neq k$.

Let $S_j = \bigcap_k S_{kj}$, then $S_j \supset R_j$. Moreover we can see that $S_iS_j \subset S_i$ holds true.

Similarly, let T_{lm} be the set of elements t of S such that $dt \varepsilon D_{km}$ where d is an element of D_{kl}, then T_{lm} has the following properties:

(i) S can be t-decomposed by d, namely

$$S = \sum_{m=1}^{q} T_{lm} \quad \text{and} \quad T_{lm} \cap T_{ln} = \phi \qquad \text{for } m \neq n,$$

(ii) $T_{lm} \supset L_m$

(iii) $D_{jl}T_{lm} = D_{jm}$ for all j.

Let $T_m = \bigcap_l T_{lm}$, then T_m has the following properties:

(i) $T_m \supset L_m$,

(ii) $T_mT_n \subset T_n$.

By C_{km} we denote $S_k \cap T_m$. It is not hard to show that $C_{km} \supset D_{km}$ and $C_{km}C_{ln} \subset C_{kn}$ hold, and therefore, the following theorem is obtained:

THEOREM 1. *D_{kl}, any one of the groups composing the kernel D, is contained respectively in semigroups C_{kl} which have no element in common. Among these semigroups,*

$$C_{km}C_{ln} \subset C_{kn} \tag{5}$$

holds true.

We shall now consider the following two conditions:

(A) $S_{kj} = S_{jj}$ for all k, j,

(B) $T_{lm} = T_{mm}$ for all l, m.

In a semigroup S, if $ax = ay$ ($xa = ya$) implies $x = y$ for every a, x, y in S, then S is called a semigroup satisfying the left (right) cancellation law.

THEOREM 2. *The condition A (B) holds true when the right (left) cancellation law is satisfied.*

Proof. Let d and d'' be elements of D_{kl} and D_{jl} respectively, then $s_{kj}d \varepsilon D_{jl}$ for $s_{kj} \varepsilon S_{kj}$. Assume now that $s_{kj}d'' \varepsilon D_{il}$ ($i \neq j$), then there exists in S_{kl} an element s_{kl} such that $s_{kj}d'' = s_{kl}d$. Nevertheless, $D_{kl}d'' = d(D_{jl}\cdot d'') = d\cdot D_{jl} = D_{kl}$. Therefore, there exists in D_{kl} an element d' such that $d'd''=d$, and we have $s_{kj}d''=s_{kl}d'd''$. By the right cancellation law, $s_{kj}=s_{kl}\cdot d'$. d and d' are elements of D_{kl}, then $s_{kl}d' \varepsilon D_{il}$, so that, $s_{kj} \varepsilon D_{il}$ is obtained. But this is contradiction since $D_{il} \subset S_{kl}$, $s_{kj} \varepsilon S_{kj}$, and $S_{kl} \cap S_{kj}=\phi$, then $s_{kj}d'' \varepsilon D_{jl}$ and we have $S_{kj} \subset S_{jj}$. On the other hand, if $s_{kk}d'' \varepsilon D_{il}$ $(i \neq k)$, there exists in S_{kl} an element s_{kl} such that $s_{kk}d''=s_{kl}d$. In the same way as above, there exists in D_{kl} an element d' such that $s_{kk}d''=s_{kl}d'd''$, and by assumption $s_{kk}=s_{kl}d'$ is obtained. But $s_{kl}d'$, together with $s_{ki}d$, is an element of D_{il}, then $s_{kk} \varepsilon D_{il}$. This is however impossible, since $s_{kk}d \varepsilon D_{kl}$. Hence we have $s_{kk}d'' \varepsilon D_{kl}$ and also $S_{kk} \subset S_{jk}$. Consequently $S_{kj}=S_{jj}$, and the condition (A) is satisfied. Analogously we can see that the condition (B) holds when the left cancellation law is satisfied. This completes the proof.

Under the condition (A), $S_j=S_{kj}$ for any k and also $S=\sum_j S_j$. Similarly under the condition (B), $T_m=T_{lm}$ for any l and $S=\sum_m T_m$. Therefore, under (A) and (B),

$$S_j=\sum_m C_{jm}, \quad T_m=\sum_k C_{km}, \quad S=\sum_{k,m} C_{km} \tag{6}$$

hold true.

It can easily be verified that S_j (T_m) is a right (left) ideal and $S_jT_m \subset C_{jm}$.

Hence we have the following

THEOREM 3. *Let S be a semigroup satisfying the conditions (A) and (B), then S is decomposed into join of left ideals T_m which have no element in common. The same holds for right ideals S_j, and among these ideals*

$$S_j \cdot T_m \subset S_j \cap T_m$$

holds true.

References

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