60. Uniform Convergence of Fourier Series. IV

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1. Let $f(x)$ be a continuous function with period 2π. After J. P. Nash [1], $f(x)$ is said to be of class $\phi(n)$ if

$$\phi(n)\int_a^b f(x+t)\cos nt\,dt=O(1)$$

uniformly for all x, n, a, b with $b-a\leqq 2\pi$.

If $\phi(n)$ is not $O(n)$, then $f(x)$ becomes constant [1] and if $\omega(1/n)\leqq 1/\phi(n)$, then $f(x)$ belongs to $\phi(n)$ class; so that we assume that

$$\phi(n)<n, \qquad \omega(1/n)\geqq 1/\phi(n).$$

We have already proved the following theorem [2]:

Theorem 1. *If $f(x)$ is of class $\phi(n)$ and is continuous with modulus of continuity $\omega(\delta)$, then there exists a positive constant C*[1] *independent of n such that*

$$|s_n(x)-f(x)|\leqq C\left[\omega\left(\frac{1}{n}\right)\log\left(Cn\frac{\theta(n)}{\phi(n)}\right)+\frac{1}{\theta(n)}\right],$$

where $\theta(n)$ is monotone increasing and $1\leqq\theta(n)\leqq\phi(n)$.

We prove here the following

Theorem 2. *Let $f(x)$ be a function not equivalent to a constant. Let $\omega(\delta)$ be the modulus of continuity of $f(x)$ and $\omega_1(\delta)$ be the integral modulus of continuity of $f(x)$.*[2] *Then,*

$$|s_n(x)-f(x)|\leqq C\,\omega\left(\frac{1}{n}\right)\log\left(Cn\frac{\omega_1(1/n)}{\omega(1/n)}\right).$$

On the other hand, concerning uniform $(C,-\alpha)$ summability of Fourier series we have proved the following theorem [3]:

Theorem 3. *If $f(x)$ is of class $\phi(n)$ and is continuous with the modulus of continuity $\omega(\delta)$, then*

$$|\sigma_n^{-\alpha}(x)-f(x)|\leqq C\left[\omega\left(\frac{1}{n}\right)^{1-\alpha}\left(\frac{n}{\phi(n)}\right)^{\alpha}+\frac{1}{n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t^2}dt\right],$$

where $0<\alpha<1$, $\sigma_n^{-\alpha}(x)$ is the n-th Cesàro mean of the Fourier series of $f(x)$ of order $-\alpha$.

From this theorem we get

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1) C is always used to denote a constant independent of n, which is different in different occurrences.

2) $$\omega_1(\delta)=\operatorname*{Max}_{0<h\leqq\delta}\int_0^{2\pi}|f(x+h)-f(x)|\,dx.$$

Theorem 4. *Let $f(x)$ be a function not equivalent to a constant, $\omega(\delta)$ be the modulus of continuity of $f(x)$ and $\omega_1(\delta)$ the integral modulus of continuity of $f(x)$. Then*

$$|\sigma_n^{-\alpha}(x)-f(x)| \leqq C\left[\omega\left(\frac{1}{n}\right)\left(n\frac{\omega_1(1/n)}{\omega(1/n)}\right)^{\alpha}+\frac{1}{n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t^2}dt\right],$$

where $0<\alpha<1$.

For the proof of these theorems we use the following theorem due to E. Hille and G. Klein [4]:[3)]

Theorem I. *Unless $f(t)$ is equivalent to a constant function,*

$$m_1(h) \leqq K\omega_1(h),$$

where $m_1(h)=\underset{0\leqq x\leqq 2\pi}{\mathrm{Max}}\int_0^h|f(x+t)|\,dt$ and K is a constant independent of h.

2. We prove the following

Lemma. *If $f(x)$ is integrable, then $f(x)$ belongs to the class $1/\omega_1(1/n)$.*

Proof. It is sufficient to prove that

$$\int_a^b f(x+t)\sin nt\,dt=O(\omega_1(1/n))$$

uniformly for all x, n, a, b with $b-a\leqq 2\pi$. We have

$$\int_a^b f(x+t)\sin nt\,dt=-\int_{a+\pi/n}^{b+\pi/n}f(x+t-\pi/n)\sin nt\,dt$$

$$=\frac{1}{2}\left[\int_a^{a+\pi/n}f(x+t)\sin nt\,dt+\int_a^b[f(x+t)-f(x+t-\pi/n)]\sin nt\,dt\right.$$
$$\left.-\int_b^{b+\pi/n}f(x+t-\pi/n)\sin nt\,dt\right],$$

hence by Theorem I

$$\left|\int_a^b f(x+t)\sin nt\,dt\right|\leqq C\Big[m_1(1/n)+\omega_1(1/n)\Big]\leqq C\omega_1(1/n).$$

Thus the lemma is proved.

By the lemma and the remark stated at the beginning of [1], $\omega(1/n)/\omega_1(1/n)\leqq Cn$ except trivial case, and then $n\omega_1(1/n)/\omega(1/n)$ becomes greater than 1 for sufficiently large n.

In Theorem 1, if we take $\phi(n)=1/\omega_1(1/n)$, then

$$|s_n(x)-f(x)|\leqq C[\omega(1/n)\log(Cn\,\theta(n)\omega_1(1/n))+1/\theta(n)].$$

Taking $\theta(n)=1/\omega(1/n)$ we get the conclusion of Theorem 2.

Theorem 4 follows from Lemma and Theorem 3, taking

$$\phi(n)=1/\omega_1(1/n).$$

3. As particular cases of Theorem 2 we get the following corollaries.

Corollary 1 (*Dini-Lipschitz* [5]). *If $f(x)$ is continuous and*

3) For a simple proof, see [8].

$$\omega(1/n)=o(1/\log n),$$

then the Fourier series of $f(x)$ converges uniformly.

Corollary 2. *If $f(x)$ is continuous and*

$$\omega(1/n)=o(1/\log\log n),\qquad \omega_1(1/n)\leqq \omega(1/n)(\log n)^k/n$$

then the Fourier series of $f(x)$ converges uniformly, where k is a positive constant.

For example, if we put

$$f(t)=1/(\log\log 1/t)^2 \qquad (0<t\leqq\pi),$$
$$f(0)=0,\quad f(t)=f(2\pi-t) \qquad (\pi<t<2\pi),$$

then the conditions of Corollary 2 are satisfied.

Corollary 3 (*de la Vallée Poussin* [5]). *If $f(x)$ belongs to the class* Lip α $(0<\alpha\leqq 1)$, *then*

$$|s_n(x)-f(x)|\leqq C\log n/n^\alpha.$$

4. We get the following corollaries to Theorem 4.

Corollary 4 (*Zygmund* [6]). *If $f(x)$ is continuous and $\omega(1/n)=o(1/n^\alpha)$, then the Fourier series of $f(x)$ is summable $(C,-\alpha)$ uniformly.*

Corollary 5. *If $f(x)$ belongs to the class* lip β *and* Lip $(1+\beta-\beta/\alpha, 1)$, *i.e.*

$$\omega(1/n)=o(1/n^\beta),\qquad \omega_1(1/n)=C/n^{1+\beta-\beta/\alpha}$$

for $0<\beta<\alpha<1$, then the Fourier series of $f(x)$ is summable $(C,-\alpha)$ uniformly.

Corollary 6 (*Hardy-Littlewood* [7]). *If $f(x)$ is continuous and of bounded variation, then the Fourier series of $f(x)$ is summable $(C,-\alpha)$ uniformly.*

For, if $f(x)$ is of bounded variation, it belongs to Lip $(1,1)$ i.e. $\omega_1(1/n)=O(1/n)$. By the continuity of $f(x)$, $\omega(1/n)=o(1)$, and hence we obtain Corollary 6 from Theorem 4.

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