59. Some Trigonometrical Series. XIII

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1. Let $f(x)$ be an integrable function with period 2π and $s_n(x)$ be the n-th partial sum of Fourier series of $f(x)$. The object of this paper is to prove that, if $f(x)$ belongs to the Lip α class, then the series[1)]

$$(1)\qquad \sum_{n=1}^{\infty} | s_n(x)-f(x) |^2/n^\beta (\log n)^\gamma$$

converges uniformly, where $\beta=1-2\alpha$ and $\gamma>1$ or >2 according as $0<\alpha<1/2$ or $1/2\leqq\alpha<1$. By the method of this paper, we can prove that if

$$| f(x+t)-f(x) | \leqq A\sqrt{t}\Big/\left(\log\frac{1}{t}\right)^\gamma$$

for $\gamma>1$, then the series

$$\sum_{n=1}^{\infty} | s_n(x)-f(x) |^2$$

converges uniformly.

Theorems of this sort are found in "Some Trigonometrical Series", VI and IX, Tôhoku Math. Journ., 1954.

2. **Theorem 1.** *If*

$$(2)\qquad | f(x+t)-f(x) | \leqq A\sqrt{t},$$

then the series

$$(3)\qquad \sum_{n=1}^{\infty} | s_n(x)-f(x) |^2/(\log n)^\gamma$$

converges uniformly for $\gamma>2$.

Proof. We have

$$(4)\qquad S_n(x)=\sum_{\nu=1}^{n} | s_\nu(x)-f(x) |^2/(\log \nu)^\gamma$$

$$=\sum_{\nu=1}^{n}\frac{1}{(\log\nu)^\gamma}\left(\frac{2}{\pi}\int_0^\pi \varphi_x(u)\frac{\sin(\nu+1/2)u}{2\sin u/2}du\right)^2$$

$$=\frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{\varphi_x(u)}{\sin u/2}\,\frac{\varphi_x(v)}{\sin v/2}\left(\sum_{\nu=1}^{n}\frac{\sin(\nu+1/2)u\sin(\nu+1/2)v}{(\log\nu)^\gamma}\right)du\,dv.$$

Now[2)]

$$\left|\sum_{\nu=1}^{n}\frac{\cos\nu x}{(\log\nu)^\gamma}\right| = \left|\sum_{\nu=1}^{1/x}+\sum_{\nu=1/x}^{n}\frac{\cos\nu x}{(\log\nu)^\gamma}\right|$$

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1) We suppose $1/(\log n)^\gamma=1$ for $n=1$.

2) A denotes an absolute constant which is not necessarily the same in different occurrences and $\Delta a_n=a_n-a_{n+1}$.

$$\leqq \sum_{\nu=1}^{1/x} \frac{1}{(\log \nu)^{\gamma}} + \frac{1}{x}\Big[\sum_{\nu=1/x}^{n} \varDelta \frac{1}{(\log \nu)^{\gamma}} + \frac{1}{(\log 1/x)^{\gamma}} + \frac{1}{(\log n)^{\gamma}}\Big]$$
$$\leqq A\Big/x\Big(\log \frac{1}{x}\Big)^{\gamma}.$$

Hence the kernel of the integral on the right of (4) is less than, in absolute value,

$$\frac{A}{|u-v|(\log 1/|u-v|)^{\gamma}} + \frac{A}{(u+v)(\log 1/(u+v))^{\gamma}}$$

and then, by (2),

$$S_n(x) \leqq A \int_0^{\pi}\int_0^{\pi} \frac{1}{\sqrt{uv}} \frac{1}{|u-v|(\log 1/|u-v|)^{\gamma}} du\, dv$$
$$= A\Big(\iint_{u>v} + \iint_{u\leqq v}\Big) = AS_{n,1}(x) + AS_{n,2}(x),$$

say. We have

$$S_{n,1}(x) \leqq \int_0^{\pi} \frac{du}{\sqrt{u}} \int_0^{u} \frac{dv}{\sqrt{v}\,(u-v)(\log 1/(u-v))^{\gamma}}$$
$$= \int_0^{\pi} \frac{du}{\sqrt{u}} \Big(\int_0^{u/2} + \int_{u/2}^{u} \frac{dv}{\sqrt{v}\,(u-v)(\log 1/(u-v))^{\gamma}}\Big)$$
$$\leqq A \int_0^{\pi} \frac{du}{u(\log 1/u)^{\gamma-1}} < \infty.$$

Since $S_{n,2}(x)$ may be similarly estimated, $S_n(x)$ is uniformly bounded, and then the series (3) converges. The uniformity of convergence follows from Abel's lemma.

3. **Theorem 2.** *Let* $0<\alpha<1/2$, $\beta=1-2\alpha$, *and* $\gamma>1$. *If*

(5) $$|f(x+t)-f(x)| \leqq At^{\alpha},$$

then the series (1) converges uniformly.

Proof. We have

$$S_n(x) = \sum_{\nu=1}^{n} |s_\nu(x)-f(x)|^2/\nu^{\beta}(\log \nu)^{\gamma}$$
$$= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \varphi_x(u)\varphi_x(v)\Big(\sum_{\nu=1}^{n}\frac{1}{\nu^{\beta}} \frac{\sin(\nu+1/2)u \sin(\nu+1/2)v}{\sin u/2 \sin v/2}\Big) du\, dv.$$

Since

$$\Big|\sum_{\nu=1}^{n} \frac{\cos \nu x}{\nu^{\beta}(\log \nu)^{\gamma}}\Big| \leqq \Big|\sum_{\nu=1}^{1/x} \frac{\cos \nu x}{\nu^{\beta}(\log \nu)^{\gamma}}\Big| + \Big|\sum_{\nu=1/x}^{n} \frac{\cos \nu x}{\nu^{\beta}(\log \nu)^{\gamma}}\Big| \leqq A\Big/x^{1-\beta}\Big(\log\frac{1}{x}\Big)^{\gamma},$$

we have, by (5),

$$S_n(x) \leqq A\int_0^{\pi}\int_0^{\pi} \frac{1}{u^{1-\alpha}v^{1-\alpha}} \frac{1}{|u-v|^{1-\beta}(\log 1/|u-v|)^{\gamma}} du\, dv$$
$$= A\Big(\iint_{u>v} + \iint_{u\leqq v}\Big) = AS_{n,1}(x) + AS_{n,2}(x),$$

say. Now

$$S_{n,1}(x) = \int_0^{\pi} du \Big(\int_0^{u/2} dv + \int_{u/2}^{u} dv\Big) \leqq A\int_0^{\pi} \frac{du}{u(\log 1/u)^{\gamma}} < \infty$$

and similarly $S_{n,2}(x)$ is also bounded, and hence the series converges and then converges uniformly.

4. Theorem 3. *Let $\alpha>1/2$, $\beta=2\alpha-1$, and $\gamma>2$. If (5) holds, then the series*

$$\sum_{n=1}^{\infty} |s_n(x)-f(x)|^2 n^\beta/(\log n)^\gamma \tag{6}$$

converges uniformly.

Proof. We have

$$S_n(x)=\sum_{\nu=1}^{n} |s_\nu(x)-f(x)|^2 \nu^\beta/(\log \nu)^\gamma$$

$$=\frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{\varphi_x(u)}{\sin u/2}\frac{\varphi_x(v)}{\sin v/2}\Big(\sum_{\nu=1}^{n}\frac{\nu^\beta}{(\log \nu)^\gamma}\sin(\nu+1/2)u\sin(\nu+1/2)v\Big)du\,dv.$$

We denote by $K_n(u,v)$ the kernel of the integral on the right. Then

$$K_n(u,v)=\frac{1}{2}\sum_{\nu=1}^{n}\frac{\nu^\beta}{(\log \nu)^\gamma}[\cos(\nu+1/2)(u-v)-\cos(\nu+1/2)(u+v)]$$

$$=\frac{1}{2}[L_n(u,v)-M_n(u,v)].$$

By Abel's lemma,

$$L_n(u,v)=\sum_{\nu=1}^{n-1}\Delta\Big(\frac{\nu^\beta}{(\log \nu)^\gamma}\Big)\frac{\sin(\nu+1)(u-v)}{\sin(u-v)/2}-\frac{n^\beta}{(\log n)^\gamma}\frac{\sin(n+1)(u-v)}{\sin(u-v)/2}$$

$$=L_{n,1}(u,v)-L_{n,2}(u,v),$$

say. Since

$$\sum_{\nu=1}^{n-1}\frac{\nu^{\beta-1}}{(\log \nu)^\gamma}\frac{\sin(\nu+1)t}{\sin t/2}=\sum_{\nu=1}^{1/t}+\sum_{\nu=1/t}^{n-1}$$

$$=O\Big(\sum_{\nu=1}^{1/t}\frac{\nu^\beta}{(\log \nu)^\gamma}+\frac{1}{t^2}\frac{1}{t^{\beta-1}(\log 1/t)^\gamma}\Big)$$

$$=O\Big(1\Big/t^{\beta+1}\Big(\log\frac{1}{t}\Big)^\gamma\Big),$$

we get $|L_{n,1}(u,v)|\leqq A\Big/t^{\beta+1}\Big(\log\frac{1}{t}\Big)^\gamma$, and then

$$\Big|\int_0^{\pi/2}\frac{\varphi_x(u)}{u}du\int_{2u}^{\pi}\frac{\varphi_x(v)}{v}L_{n,1}(u,v)\,dv\Big|$$

$$\leqq A\int_0^{\pi/2}\frac{du}{u^{1-\alpha}}\int_{2u}^{\pi}\frac{dv}{v^{1-\alpha}(v-u)^{\beta+1}(\log 1/(v-u))^\gamma}$$

$$\leqq A\int_0^{\pi/2}\frac{du}{u^{1-\alpha}}\int_{2u}^{\pi}\frac{dv}{v^{1+\alpha}(\log 1/v)^\gamma}\leqq A\int_0^{\pi}\frac{du}{u(\log 1/u)^\gamma}<\infty.$$

On the other hand,

$$\int_0^{\pi/2}\frac{\varphi_x(u)}{u}du\int_{2u}^{\pi}\frac{\varphi_x(v)}{v}L_{n,2}(u,v)\,dv$$

$$=\int_0^{1/n}du\int_{2u}^{\pi}dv+\int_{1/n}^{\pi/2}du\int_{2u}^{\pi}dv=I+J,$$

say, where

$$|I| \leqq \frac{n^\beta}{(\log n)^\gamma} \int_0^{1/n} \frac{du}{u^{1-\alpha}} \int_{2u}^{\pi} \frac{dv}{v^{1-\alpha}(v-u)}$$
$$\leqq A \frac{n^\beta}{(\log n)^\gamma} \int_0^{1/n} \frac{du}{u^{2(1-\alpha)}} \leqq \frac{A}{(\log n)^\gamma},$$

and, since

$$\left| \int_{2u}^{\pi} \frac{\varphi_x(v)}{v} \frac{\sin n(u-v)}{(u-v)} dv \right|$$
$$\leqq \int_{2u}^{\pi} \left| \frac{\varphi_x(v)}{v(v-u)} - \frac{\varphi_x(v+\pi/n)}{(v+\pi/n)(v+\pi/n-u)} \right| dv + \int_{2u}^{2u+\pi/n} \frac{|\varphi_x(v)|}{v(v-u)} du + \frac{A}{n}$$
$$\leqq A \int_{2u}^{\pi} \frac{(\pi/n)^\alpha}{v^2} dv + \frac{A}{n} \int_{2u}^{\pi} \frac{dv}{v^{3-\alpha}} + \frac{A}{nu^{2-\alpha}} + \frac{A}{n} \leqq \frac{A}{n^\alpha u} + \frac{A}{nu^{2-\alpha}},$$

we get

$$|J| \leqq \frac{An^\beta}{(\log n)^\gamma} \int_{1/n}^{\pi/2} \frac{du}{u^{1-\alpha}} \left(\frac{1}{n^\alpha u} + \frac{1}{nu^{2-\alpha}} \right)$$
$$\leqq \frac{An^\beta}{(\log n)^\gamma} \left(\frac{1}{n^\alpha} \cdot n^{1-\alpha} + \frac{1}{n} \cdot n^{2-2\alpha} \right) \leqq \frac{A}{(\log n)^\gamma}.$$

Thus $I+J=o(1)$. Since similar estimation holds for the integral with kernel $M_n(u, v)$, it remains to prove the boundedness of the integral

$$\int_0^{\pi} \frac{\varphi_x(x)}{\sin v/2} dv \int_{v/2}^{v} \frac{\varphi_x(u)}{\sin u/2} K_n(u, v)\, du$$

which is

$$(7) \quad \int_0^{\pi} \frac{\varphi_x(v)}{\sin v/2} dv \int_{v/2}^{v} \frac{\varphi_x(u)}{\sin u/2} \left(\sum_{\nu=1}^{n} \frac{\nu^\beta}{(\log \nu)^\gamma} \sin(\nu+1/2)u \sin(\nu+1/2)v \right) du$$
$$= \sum_{\nu=1}^{n} \frac{\nu^\beta}{(\log \nu)^\gamma} \int_0^{\pi} \frac{\varphi_x(v)}{\sin v/2} \sin(\nu+1/2)v\, dv \int_{v/2}^{v} \frac{\varphi_x(u)}{\sin u/2} \sin(\nu+1/2)u\, du.$$

We have, putting $\mu=\nu+1/2$,

$$(8) \quad \int_{v/2}^{v} \frac{\varphi_x(u)}{\sin u/2} \sin \mu u\, du = \frac{1}{2} \int_{v/2}^{v} \left(\frac{\varphi_x(u)}{\sin u/2} - \frac{\varphi_x(u-\pi/\mu)}{\sin(u-\pi/\mu)/2} \right) \sin \mu u\, du$$
$$+ \frac{1}{2} \int_{v/2}^{v/2+\pi/\mu} \frac{\varphi_x(u-\pi/\mu)}{\sin(u-\pi/\mu)/2} \sin \mu u\, du - \frac{1}{2} \int_{v}^{v+\pi/\mu} \frac{\varphi_x(u-\pi/\mu)}{\sin(u-\pi/\mu)/2} \sin \mu u\, du.$$

Substituting this into the double integral of the right of (6), inverting the order of integration and using the device similar to (8), we can see that the double integral is of order $O(\log \nu/\nu^{2\gamma})$. Hence (7) is bounded for $\gamma > 2$. Thus the theorem is proved.