77. *Integrations on the Circle of Convergence and the Divergence of Interpolations. I*

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Let the points

$$\text{(P)}\qquad \begin{cases} z_1^{(1)} \\ z_1^{(2)},\ z_2^{(2)} \\ z_1^{(3)},\ z_2^{(3)},\ z_3^{(3)} \\ \cdots\cdots \\ z_1^{(n)},\ z_2^{(n)},\ z_3^{(n)},\ \ldots,\ z_n^{(n)} \\ \cdots\cdots \end{cases}$$

which do not lie exterior to the unit circle $C: |z|=1$, satisfy the condition that the sequence of

$$\frac{w_n(z)}{z^n}=(z-z_1^{(n)})(z-z_2^{(n)})\cdots(z-z_n^{(n)})/z^n$$

converges to a function $\lambda(z)$ single valued, analytic, and non-vanishing for z exterior to C, and uniformly for any finite closed set exterior to C, that is

$$\text{(C)}\qquad \lim_{n\to\infty}\frac{w_n(z)}{z^n}=\lambda(z)\neq 0 \quad \text{for} \quad |z|>1.$$

Let $f(z)$ be a function single valued and analytic within the circle $C_\rho: |z|=\rho>1$ but not analytic on C_ρ. Then the sequence of polynomials $P_n(z;f)$ of respective degrees n which interpolate to $f(z)$ in all the zeros of $w_{n+1}(z)$ is known to be

$$\text{(I)}\qquad P_n(z;f)=\frac{1}{2\pi i}\int_{C_R}\frac{w_{n+1}(t)-w_{n+1}(z)}{w_{n+1}(t)}\,\frac{f(t)}{t-z}\,dt, \quad (1<R<\rho).$$

It is known that the sequence of polynomials $P_n(z;f)$ converges to $f(z)$ throughout the interior of the circle C_ρ, and uniformly for any closed set interior to C_ρ. But the divergence of $P_n(z;f)$ at every point exterior to C_ρ is not established in general.

This problem is seen in the paper by Walsh: *The divergence of sequences of polynomials interpolating in roots of unity;* Bulletin of the American Mathematical Society, 1936, Vol. 42, p. 715. And that is treated in the following papers by the author.

T. Kakehashi: *On the convergence-region of interpolation polynomials;* Journal of the Mathematical Society of Japan, 1955, Vol. 7, p. 32.

T. Kakehashi: *The divergence of interpolations.* I, II, III; Proceedings of the Japan Academy, 1954, Vol. 30, Nos. 8, 9, and 10.

In this paper, we consider a certain type of integrations on the convergence-circle of a function, which belongs to a certain class of functions, and consider the divergence properties of $P_n(z; f)$ at every point exterior to the convergence-circle.

1. Let $F(\theta); 0 \leqq \theta \leqq 2\pi$ be a complex valued function with the bounded variation (not necessarily periodic). Then the function

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho e^{i\theta}}{\rho e^{i\theta}-z}dF(\theta), \qquad |z|<\rho$$

is single valued and analytic within the circle C_ρ: $|z|=\rho$.

Definition 1. *Let K_ρ $(\rho>0)$ be denoted by the class of functions $f(z)$ which satisfy the conditions*

$$f(z)=\sum_{n=0}^{\infty}c_n\left(\frac{z}{\rho}\right)^n=\frac{1}{2\pi}\int_0^{2\pi}\frac{\rho e^{i\theta}}{\rho e^{i\theta}-z}dF(\theta) \qquad (\rho>0), \tag{1.1}$$

$$c_n=\frac{1}{2\pi}\int_0^{2\pi}e^{-in\theta}dF(\theta); \qquad n=0, 1, 2, \dots, \tag{1.2}$$

and

$$\overline{\lim}_{n\to\infty}|c_n|>0, \tag{1.3}$$

where $F(\theta)$ is a complex valued function with the bounded variation and is normalized by

$$F(0)=0, \quad F(\theta-0)=F(\theta). \tag{1.4}$$

It is clear that a function which belongs to K_ρ is single valued and analytic within the circle $C_\rho: |z|=\rho$ but not analytic on C_ρ, and that, in the power series $\sum c_n\left(\frac{z}{\rho}\right)^n$, the coefficients c_n satisfy $0<\overline{\lim}_{n\to\infty}|c_n|<\infty$. And the function $F(\theta)$ in (1.2) can not be absolutely continuous by the *Riemann-Lebesgue theorem*. It is to be noticed that the Fourier-Stieltjes coefficients c_{-n} with negative suffixes of $F(\theta)$ is not considered.

For example, if $F(\theta)$ is a step function, $f(z)$ is a function with poles of first order on the circle C_ρ.

Let $f(z)$ be the function which belongs to the class K_ρ defined by (1.1) and $\varphi(z)$ be a function single valued and analytic on C_ρ, and be defined by the Laurent's series

$$\varphi(z)=\sum_{n=-\infty}^{\infty}a_n z^n. \tag{1.5}$$

We can not define in general the integral

$$\int_{C_\rho}\varphi(t)f(t)\,dt$$

in ordinary sense. In this case, we define the finite part of the above integral by

$$\frac{pf.}{2\pi i}\int_{C_\rho}\varphi(t)f(t)\,dt=\frac{1}{2\pi}\int_0^{2\pi}\varphi^*(\rho e^{i\theta})\,\rho e^{i\theta}\,dF(\theta), \tag{1.6}$$

where $\varphi^*(z)$ is the principal part of the Laurent's series (1.5), that is

$$\varphi^*(z)=\sum_{n=1}^{\infty}a_{-n}z^{-n}. \tag{1.7}$$

Lemma 1.1. *Let $f(z)$ be a function which belongs to the class K_ρ, and $\varphi(z)$ be a function single valued and analytic on and between the two circles $C_\rho: |z|=\rho$ and $C_R: |z|=R<\rho$. Then*

$$(1.8) \qquad pf.\int_{C_\rho} \varphi(t) f(t)\, dt = \int_{C_R} \varphi(t) f(t)\, dt.$$

By *Cauchy's theorem*, we have

$$\frac{1}{2\pi i}\int_{C_R} \varphi(t) f(t)\, dt = \frac{1}{2\pi i}\int_{C_R} \varphi^*(t) f(t)\, dt = \sum_{n=0}^{\infty} c_n \alpha_{-n-1}\rho^{-n},$$

where $\varphi^*(t)$, c_n, and α_{-n-1} are defined respectively by (1.7), (1.1), and (1.7). It is clear that the last side is convergent. The left-side of (1.8) is also

$$\frac{pf.}{2\pi i}\int_{C_\rho} \varphi(t) f(t) dt = \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=0}^{\infty} \alpha_{-n-1}\rho^{-n} e^{-ni\theta}\, dF(\theta) = \sum_{n=0}^{\infty} c_n \alpha_{-n-1}\rho^{-n}.$$

Thus the lemma is established.

Lemma 1.2. *Let $f(z)$ be a function which belongs to the class K_ρ, and a function $\varphi(z)$ single valued and analytic on C_ρ be non-vanishing on C_ρ. Then*

$$(1.9) \qquad \overline{\lim}_{n\to\infty} \left| \frac{\rho^n pf.}{2\pi i}\int_{C_\rho} \varphi(t) f(t)\, t^{-n-1}\, dt \right| > 0.$$

The function $1/\varphi(z)$ which is single valued and analytic on C_ρ can be expanded into the Laurent's series

$$1/\varphi(z) = \sum_{n=-\infty}^{\infty} \beta_n \left(\frac{z}{\rho}\right)^n = \sum_{n=-\infty}^{\infty} \beta_n e^{ni\theta},$$

which is absolutely convergent on C_ρ. The function $\varphi(z) f(z)$ is also expanded into the Laurent's series

$$\varphi(z) f(z) = \sum_{-\infty}^{\infty} \gamma_n \left(\frac{z}{\rho}\right)^n.$$

Then the function $f(z)$ can be expanded into

$$(1.10) \qquad f(z) = \sum_{n=0}^{\infty} c_n \left(\frac{z}{\rho}\right)^n = \sum_{n=0}^{\infty} \left(\sum_{p=-\infty}^{\infty} \gamma_p \beta_{n-p}\right) \left(\frac{z}{\rho}\right)^n.$$

If we assume

$$\lim_{n\to\infty} \frac{\rho^n pf.}{2\pi i}\int_{C_\rho} \varphi(t) f(t) t^{-n-1}\, dt = 0,$$

we have $\lim_{n\to\infty}\gamma_n = 0$ by lemma 1.1. Let $M=\max|\gamma_n|$, then as n tends to infinity

$$|c_n| = \left|\sum_{p=-\infty}^{\infty} \gamma_p \beta_{n-p}\right| \leqq M \sum_{p=-\infty}^{\left[\frac{n}{2}\right]} |\beta_{n-p}| + \max_{p>\frac{n}{2}} |\gamma_p| \sum_{p=\left[\frac{n}{2}\right]+1}^{\infty} |\beta_{n-p}|$$

$$\leqq M \sum_{q=n-\left[\frac{n}{2}\right]}^{\infty} |\beta_q| + \max_{p>\frac{n}{2}} |\gamma_p| \sum_{q=-\infty}^{\infty} |\beta_q| \to 0$$

by the absolutely convergence of $\sum \beta_n e^{ni\theta}$ and $\gamma_n \to 0$ as $n\to\infty$, which contradicts the condition of $f(z)$. Thus the lemma has been proved.

Lemma 1.3. *Let $f(z)$ be a function of the class K_ρ and the sequence of $\varphi_n(z): n=1, 2, \dots$ single valued and analytic on C_ρ tend to zero uniformly on C_ρ as $n \to \infty$. Then*

$$\lim_{n\to\infty} \frac{\rho^n pf.}{2\pi i} \int_{C_\rho} \varphi_n(t) f(t)\, t^{-n-1}\, dt = 0. \tag{1.11}$$

Let $\psi_n(t); n=1, 2, \dots$ be the principal parts respectively of $\varphi_n(t)t^{-n-1}$. It is clear that $\psi_n(t)\rho^n$ tends to zero on C_ρ as $n \to \infty$ by the condition of $\varphi_n(t)$.

Hence

$$\frac{\rho^n pf.}{2\pi i} \int_{C_\rho} \varphi_n(t)\, f(t)\, t^{-n-1}\, dt = \frac{1}{2\pi} \int_0^{2\pi} \psi_n(\rho e^{i\theta})\, \rho e^{i\theta}\, dF(\theta)$$

tends to zero as $n \to \infty$ by the boundedness of $|dF(\theta)|$. Thus the lemma is established.

2. In this paragraph, we consider the divergence of interpolation polynomials of a function which belongs to K_ρ.

Theorem 1. *Let $f(z)$ be a function which belongs to the class $K_\rho(\rho > 1)$ and (P) be the points set which satisfies the condition (C). Then the sequence of polynomials $P_n(z; f)$ of respectively degrees n found by interpolation to $f(z)$ in all the zeros of $w_{n+1}(z)$ diverges at every point exterior to C_ρ. Moreover, we have*

$$\overline{\lim}_{n\to\infty} \left| \left(\frac{\rho}{z}\right)^n P_n(z; f) \right| > 0 \quad \text{for} \quad |z| > \rho > 1. \tag{2.1}$$

In the proof of this theorem, it is convenient to have the

Lemma 2.1. *Let $f(z)$ be the function of $K_\rho(\rho > 1)$, $\lambda(z)$ be a function single valued and analytic exterior to the unit circle $C: |z|=1$ with positive modulus. Let $S_n(z; f)$ be the sequence of functions defined by*

$$S_n(z; f) = \frac{pf.}{2\pi i} \int_{C_\rho} \frac{\lambda(t)t^{n+1} - \lambda(z)z^{n+1}}{\lambda(t)t^{n+1}} \frac{f(t)}{t-z}\, dt. \tag{2.2}$$

Then

$$\overline{\lim}_{n\to\infty} \left| \left(\frac{\rho}{z}\right)^n S_n(z; f) \right| > 0 \quad \text{for} \quad |z| > \rho > 1. \tag{2.3}$$

If $\lambda(z) \equiv 1$, $S_n(z; f)$ are partial sums of the power series of $f(z)$.

Now we shall prove the lemma. In (2.2), for a fixed point z exterior to C_ρ,

$$\left(\frac{\rho}{z}\right)^{n+1} \frac{pf.}{2\pi i} \int_\rho \frac{f(t)}{t-z}\, dt$$

converges clearly to zero as $n \to \infty$. And

$$\left(\frac{\rho}{z}\right)^{n+1} \frac{pf.}{2\pi i} \int_{C_\rho} \frac{\lambda(z)z^{n+1}}{\lambda(t)t^{n+1}} \frac{f(t)}{t-z}\, dt = \rho^{n+1}\lambda(z) \frac{pf.}{2\pi i} \int_{C_\rho} [\lambda(t)(t-z)]^{-1} f(t)\, t^{-n-1}\, dt$$

does not tend to zero as $n \to \infty$ by lemma 1.2. Now the relation (2.3) follows at once. Thus the lemma is established.

Now we are in a position to prove the theorem. The sequence of polynomials $P_n(z;f)$ of respective degrees n which interpolate to $f(z)$ in all the zeros of $w_{n+1}(z)$ is given by

$$(2.4)\qquad P_n(z;f)=\frac{pf.}{2\pi i}\int_{C_\rho}\frac{w_{n+1}(t)-w_{n+1}(z)}{w_{n+1}(t)}\frac{f(t)}{t-z}dt.$$

Then we have

$$S_n(z;f)-P_n(z;f)=\frac{pf.}{2\pi i}\int_{C_\rho}\left\{\frac{w_{n+1}(z)}{w_{n+1}(t)}-\frac{\lambda(z)z^{n+1}}{\lambda(t)t^{n+1}}\right\}\frac{f(t)}{t-z}dt$$

for z exterior to the unit circle C. And we have

$$\left(\frac{\rho}{z}\right)^{n+1}\left\{\frac{w_{n+1}(z)}{w_{n+1}(t)}-\frac{\lambda(z)z^{n+1}}{\lambda(t)t^{n+1}}\right\}=\left(\frac{\rho}{t}\right)^{n+1}\left\{\frac{w_{n+1}(z)}{w_{n+1}(t)}\frac{t^{n+1}}{z^{n+1}}-\frac{\lambda(z)}{\lambda(t)}\right\}$$

$$=\left(\frac{\rho}{t}\right)^{n+1}\left\{\frac{w_{n+1}(z)}{z^{n+1}}\left(\frac{t^{n+1}}{w_{n+1}(t)}-\frac{1}{\lambda(t)}\right)+\frac{1}{\lambda(t)}\left(\frac{w_{n+1}(z)}{z^{n+1}}-\lambda(z)\right)\right\}$$

$$\equiv\left(\frac{\rho}{t}\right)^{n+1}\varphi_n(t,z);\qquad n=1,2,\ldots,$$

where $\varphi_n(t,z)$ is the sequence of functions of t, for any fixed z exterior to C_ρ, single valued and analytic on C_ρ and tends to zero as $n\to\infty$ by the condition

$$(2.5)\qquad \lim_{n\to\infty}\frac{w_n(z)}{z^n}=\lambda(z)\neq 0$$

uniformly for any finite closed points set exterior to C; $|z|=1$.

Now the relation

$$(2.6)\ \lim_{n\to\infty}\left(\frac{\rho}{z}\right)^n\left\{S_n(z;f)-P_n(z;f)\right\}=\lim_{n\to\infty}\frac{\rho^n pf.}{2\pi i}\int_{C_\rho}\varphi_n(t,z)f(t)t^{-n-1}dt=0$$

follows at once by lemma 1.3. Now we can verify from (2.3) and (2.6)

$$\overline{\lim}_{n\to\infty}\left|\left(\frac{\rho}{z}\right)^n P_n(z;f)\right|>0$$

for z exterior to C_ρ. That is, the sequence of polynomials $P_n(z;f)$ diverges with the order $\left|\frac{z}{\rho}\right|^n$ as $S_n(z;f)$ diverges. Thus the theorem has been established.