73. *Cycles and Multiple Integrals on Abelian Varieties*

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(Comm. by Z. SUETUNA, M.J.A., June 13, 1955)

In the present note we shall show some relations between cycles and multiple integrals which are similar to the known relations between divisors and simple integrals on abelian varieties.

Let **A** be an abelian variety of dimension n defined by a period matrix

$$\Omega=\begin{pmatrix}\omega_{11}, \ldots, \omega_{12n} \\ \vdots \qquad \vdots \\ \omega_{n1}, \ldots, \omega_{n2n}\end{pmatrix}.$$

We denote by $Z_1, \ldots, Z_{2n}$ the cycles of dimension are on **A** induced by vectors ${}^t(\omega_{11}, \ldots, \omega_{n1}), \ldots, {}^t(\omega_{12n}, \ldots, \omega_{n2n})$ respectively. We denote by $Z_{i_1\ldots i_r}$ $i_1 < \cdots < i_r$ the cycles of dimension r on **A** induced by *parallelotopes* of dimension r

$$\begin{pmatrix}\omega_{1i_1} \ldots \omega_{1i_r} \\ \vdots \qquad \vdots \\ \omega_{ni_1} \ldots \omega_{ni_r}\end{pmatrix}$$

respectively. Denoting by $dz_1, \ldots, dz_n$ the differentials of the first kind associated with the period systems $(\omega_{11}, \ldots, \omega_{12n}), \ldots, (\omega_{n1}, \ldots, \omega_{n2n})$ respectively, we mean by $\Omega^{(p,q)}$ the period matrix of *r-ple* differentials $dz_{j_1}\ldots dz_{j_p} d\bar{z}_{k_1}\ldots d\bar{z}_{k_q}$ whose type (p, q) satisfies $p-q=r$ mod. 4 with the period cycles $Z_{i_1\ldots i_r}$, where $j_1 < \cdots < j_p$, $k_1 < \cdots < k_q$, $i_1 < \cdots < i_r$.

We put

$$(dz_{j_1}\ldots dz_{j_p} dz_{k_1}\ldots dz_{k_q})^\dagger = \varepsilon^{1\ldots n}_{j_1\ldots j_n} \varepsilon^{1\ldots n}_{k_1\ldots k_n} dz_{k_{q+1}}\ldots dz_{k_n} d\bar{z}_{j_{p+1}}\ldots d\bar{z}_{j_n}$$
$$Z_{i_1}{}^\dagger{}_{\ldots i_r} = \varepsilon^{1\ldots 2n}_{i_1\ldots i_{2n}} Z_{i_{r+1}\ldots i_{2n}}.$$

We assume that the orders of suffixes of matrix elements in $\Omega^{(p,q)}$, $\Omega^{(n-q,n-p)}$ are chosen in such a way that

$dz_{j_1}\ldots dz_{j_p} d\bar{z}_{k_1}\ldots d\bar{z}_{k_q}$, $Z_{i_1\ldots i_r}$ corresponds to
$(dz_{j_1}\ldots dz_{j_p} d\bar{z}_{k_1}\ldots d\bar{z}_{k_q})^\dagger$, $Z_{i_1}{}^\dagger{}_{\ldots i_r}$ respectively.

We denote by σ the homomorphism from $\mathbf{A}\times\mathbf{A}$ onto **A** such that $\sigma(x, y)=x+y$ and denote by σ^* its dual mapping. We mean by $I(X\cdot Y)$ the Kronecker index of $X\cdot Y$.

Lemma 1. Let C be a cycle of dimension $2(n-r)$. Then

$$I(Z_{i_1\ldots i_r}\cdot\sigma(C\times Z_{j_1\ldots j_r}))=I(C\cdot\sigma(Z_{i_1\ldots i_r}\times Z_{j_1\ldots j_r})).$$

Proof. Putting

$$C=\sum_{l_1<\cdots<l_{2n-2r}} a_{l_1\ldots l_{2n-2r}} Z_{l_1\ldots l_{2n-2r}},$$

we get

$$\sigma(C\times Z_{j_1\cdots j_r})=\sum_{l_1<\cdots<l_{2n-2r}}\sigma(a_{l_1\cdots l_{2n-2r}}Z_{l_1\cdots l_{2n-2r}}\times Z_{j_1\cdots j_r})$$
$$=\sum_{l_1<\cdots<l_{2n-2r}}a_{l_1\cdots l_{2n-2r}}\varepsilon^{h_1\cdots h_{2n-r}}_{l_1\cdots l_{2n-2r}j_1\cdots j_r}Z_{h_1\cdots h_{2n-r}}.$$

Hence

$$I(Z_{i_1\cdots i_r}\cdot\sigma(C\times Z_{j_1\cdots j_r}))=I(\sum_{l_1<\cdots<l_{2n-2r}}a_{l_1\cdots l_{2n-2r}}\varepsilon^{h_1\cdots h_{2n-r}}_{l_1\cdots l_{2n-2r}j_1\cdots j_r}Z_{i_1\cdots i_r}Z_{h_1\cdots h_{2n-r}})$$
$$=\sum_{l_1<\cdots<l_{2n-2r}}a_{l_1\cdots l_{2n-2r}}\varepsilon^{h_1\cdots h_{2n-r}}_{l_1\cdots l_{2n-2r}j_1\cdots j_r}\varepsilon^{1\cdots 2n}_{i_1\cdots i_r h_1\cdots h_{2n-r}}$$
$$=\sum_{l_1<\cdots<l_{2n-2r}}a_{l_1\cdots l_{2n-2r}}\varepsilon^{1\cdots 2n}_{i_1\cdots i_r l_1\cdots l_{2n-2r}j_1\cdots j_r}.$$

On the other hand

$$I(C\cdot\sigma(Z_{i_1\cdots i_r}\times Z_{j_1\cdots j_r}))=I(\sum_{l_1<\cdots<l_{2n-2r}}a_{l_1\cdots l_{2n-2r}}Z_{l_1\cdots l_{2n-2r}}\varepsilon^{m_1\cdots m_{2r}}_{i_1\cdots i_r j_1\cdots j_r}Z_{m_1\cdots m_{2r}})$$
$$=\sum_{l_1<\cdots<l_{2n-2r}}a_{l_1\cdots l_{2n-2r}}\varepsilon^{m_1\cdots m_{2r}}_{i_1\cdots i_r j_1\cdots j_r}\varepsilon^{1\cdots 2n}_{l_1\cdots l_{2n-2r}m_1\cdots m_{2r}}$$
$$=\sum_{l_1<\cdots<l_{2n-2r}}a_{l_1\cdots l_{2n-2r}}\varepsilon^{1\cdots 2n}_{i_1\cdots i_r l_1\cdots l_{2n-2r}j_1\cdots j_r}.$$

This proves our Lemma.

Theorem 1. Let C be a cycle of type $(n-r, n-r)$ on **A**. Then there exist matrices $\Lambda_0(C), \Lambda_1(C), \dots, \Lambda_{[\frac{r}{2}]}(C)$ such that

$$\begin{pmatrix}\Lambda_0(C) & & & \\ & \Lambda_1(C) & & \\ & & \ddots & \\ & & & \Lambda_{[\frac{r}{2}]}\end{pmatrix}\begin{pmatrix}\Omega^{(r,0)}\\ \Omega^{(r-2,2)}\\ \vdots \\ \Omega^{(r-2[\frac{r}{2}],\,2[\frac{r}{2}])}\end{pmatrix}$$
$$=\begin{pmatrix}\Omega^{(n,n-r)}\\ \Omega^{(n-2,n-r+2)}\\ \vdots\\ \Omega^{(n-2[\frac{r}{2}],\,n-r+2[\frac{r}{2}])}\end{pmatrix}\Big(I(C\cdot\sigma(Z_{i_1\cdots i_r}\times Z_{j_1\cdots j_r}))\Big).$$

Moreover $\Lambda_0(C)=0$ implies $\Lambda_1(C)=\cdots=\Lambda_{[\frac{r}{2}]}(C)=0$. If C is not of type $(n-r, n-r)$, then there exists no such a matrix $\Lambda_2(C)$.

Proof.

$$\Omega^{(n-2\nu,n-r+2\nu)}(I(C\cdot\sigma(Z_{i_1\cdots i_r}\times Z_{j_1\cdots j_r})))$$
$$=\Big(\int_{z^{\dagger}_{j_1\cdots j_r}}dz_{l_1}\dots dz_{l_{n-2\nu}}d\bar{z}_{k_1}\dots d\bar{z}_{k_{n-r+2\nu}}\Big)\Big(I(Z_{i_1\cdots i_r}\cdot\sigma(C\times Z_{h_1\cdots h_r}))\Big)$$
$$=\Big(\int_{\sigma(C\times Z_{j_1\cdots j_r})}dz_{l_1}\dots dz_{l_{n-2\nu}}d\bar{z}_{k_1}\dots d\bar{z}_{k_{n-r+2\nu}}\Big)$$
$$=\Big(\int_{C\times Z_{j_1\cdots j_r}}\sigma^*(dz_{l_1}\dots dz_{l_{n-2\nu}}d\bar{z}_{k_1}\dots d\bar{z}_{k_{n-r+2\nu}})\Big)$$
$$=\Big(\int_{C\times Z_{j_1\cdots j_r}}(du_{l_1}+dv_{l_1})\cdots(du_{l_{n-2\nu}}+dv_{l_{n-2\nu}})$$
$$(d\bar{u}_{k_1}+d\bar{v}_{k_1})\cdots(d\bar{u}_{k_{n-r+2\nu}}+d\bar{v}_{k_{n-r+2\nu}})\Big),$$

where $\{du_1,\dots,du_n\}$, $\{dv_1,\dots,dv_n\}$ are the systems of the differentials on the first and the second component of $\mathbf{A}\times\mathbf{A}$ corresponding to $\{dz_1,\dots,dz_n\}$ on **A**.

Since C is of type $(n-r, n-r)$,

$$\int_C du_{l_1}\dots du_{l_{n-r\pm s}} d\bar{u}_{k_1}\dots d\bar{u}_{k_{n-r\mp s}}=0.$$

Putting

$$b_{i_1\dots i_{n-r}, j_1\dots j_{n-r}}=\int_C du_{i_1}\dots du_{i_{n-r}} d\bar{u}_{j_1}\dots d\bar{u}_{j_{n-r}},$$

we get

$$\int_{C\times Z_{j_1\dots j_r}} (du_{l_1}+dv_{l_1})\cdots(du_{l_{n-2\nu}}+dv_{l_{n-2\nu}})(d\bar{u}_{k_1}+d\bar{v}_{k_1})\cdots(d\bar{u}_{k_{n-r+2\nu}}+d\bar{v}_{k_{n-r+2\nu}})$$

$$=\sum_{\substack{\{\alpha_1\dots\alpha_{2n-2\nu}\}=\{l_1\dots l_{2n-2\nu}\}\\ \{\beta_1\dots\beta_{2n-2\nu}\}=\{k_1\dots k_{2n-2\nu}\}}} \pm\ b_{\alpha_1\dots\alpha_{n-r}, \beta_1\dots\beta_{n-r}}\int_{Z_{j_1\dots j_r}} dv_{\alpha_{n-r+1}}\dots dv_{\alpha_{n-2\nu}} d\bar{v}_{\beta_{n-r+1}}\dots d\bar{v}_{\beta_{n-r+2\nu}}.$$

This shows that

$$\Omega^{(n-2\nu, n-r+2\nu)}(I(C\sigma(Z_{i_1\dots i_r}\times Z_{j_1\dots j_r})))=\Lambda_\nu(C)\,\Omega^{(r-2\nu, 2\nu)}$$

with a matrix $\Lambda_\nu(C)$ whose elements are $b_{l_1\dots l_{n-r}, k_1\dots k_{n-r}}$ or zero.
On the other hand

$$(du_1+dv_1)\cdots(du_n+dv_n)(d\bar{u}_{k_1}+d\bar{v}_{k_1})\cdots(d\bar{u}_{k_{n-r}}+d\bar{v}_{k_{n-r}})$$

$$=\sum_{\{\alpha_1\dots\alpha_n\}=\{1\dots n\}} \pm\ b_{\alpha_1\dots\alpha_{n-r}, k_1\dots k_{n-r}}\int_{Z_{j_1\dots j_r}} dv_{\alpha_{n-r+1}}\dots dv_{\alpha_n}.$$

Hence all $b_{l_1\dots l_{n-r}, k_1\dots k_{n-r}}$ appear in $\Lambda_0(C)$. This shows that $\Lambda_0(C)=0$ implies $\Lambda_1(C)=\cdots=\Lambda_{[\frac{r}{2}]}(C)=0$: If C is not of type $(n-r, n-r)$, then there exists a differential

$dz_{l_1}\dots dz_{l_{n-r+s}} d\bar{z}_{k_1}\dots d\bar{z}_{k_{n-r-s}}$ with a non-zero period on C.

On the other hand

$$\int_{C\times Z_{j_1\dots j_r}} (du_1+dv_1)\cdots(du_n+dv_n)(d\bar{u}_{k_1}+d\bar{v}_{k_1})\cdots(d\bar{u}_{k_{n-r}}+d\bar{v}_{k_{n-r}})$$

$$=\pm\int_C du_{l_1}\dots du_{l_{n-r+s}} d\bar{u}_{k_1}\dots d\bar{u}_{k_{n-r-s}}$$

$$\int_{Z_{j_1\dots j_r}} dv_{l_{n-r+s+1}}\dots dv_{l_n} d\bar{v}_{k_{n-r-s+1}}\dots d\bar{v}_{k_{n-r}}\pm\dots\ .$$

The type of $dv_{l_{n-r+s+1}}\dots dv_{l_n} d\bar{v}_{k_{n-r-s+1}}\dots dv_{k_{n-r}}$ is $(r-s, s)$.

This proves the last assertion.

Lemma 2. Let P be a principal matrix of Ω and let $P^{(r)}$ be the r-th compound matrix of P. Then there exists a matrix B such that

$$B\,\Omega^{(r,0)}=\Omega^{(n,n-r)}P^{(r)}.$$

Proof. Since P is a principal matrix of Ω, there exists a non-singular matrix B_1 such that

$$\begin{pmatrix}\Omega\\ \bar{\Omega}\end{pmatrix}P^{-1}\,{}^t\overline{\begin{pmatrix}\Omega\\ \bar{\Omega}\end{pmatrix}}=\begin{pmatrix}B_1 & 0\\ 0 & \bar{B}_1\end{pmatrix}.$$

On the other hand

$$\begin{pmatrix}\Omega\\ \bar{\Omega}\end{pmatrix}{}^t\overline{\begin{pmatrix}\Omega^{(n,n-1)}\\ \bar{\Omega}^{(n,n-1)}\end{pmatrix}}=\left|\begin{pmatrix}\Omega\\ \bar{\Omega}\end{pmatrix}\right|E.$$

Hence we get

$$\begin{pmatrix} \Omega^{(10)} \\ \overline{\Omega^{(10)}} \end{pmatrix} P^{-1} \begin{pmatrix} \Omega^{(n,n-1)} \\ \overline{\Omega^{(n,n-1)}} \end{pmatrix}^{-1} = \begin{pmatrix} B_1 & 0 \\ 0 & \bar{B}_1 \end{pmatrix}.$$

Taking r-th compound of the both sides, we get

$$\Omega^{(r,0)} P^{(r)-1} = B_1^{(r)} \Omega^{(n,n-r)}.$$

From Theorem 1 and Lemma 2 we have

Theorem 2. Let **A** be an abelian variety of dimension n defined by a period matrix Ω. Let P be principal matrix of Ω and let $P^{(r)}$ be the r-th compound matrix of P. Then the module of homology classes of cycles of type $(n-r, n-r)$ is isomorphic with the module of all rational matrices M satisfying

1) $\Omega^{(r,0)} M = \Lambda \Omega^{(r,0)}$ with a matrix Λ,

2) $P^{(r)} M = (b_{i_1 \dots i_r, j_1 \dots j_r})$ is integral,

3) $b_{i_1 \dots i_r, j_1 \dots j_r}$ $i_1 < \cdots < i_r$, $j_1 < \cdots < j_r$ are skew symmetric on $\{i_1 i_2 \dots i_r\ j_1 \dots j_r\}$.