123. *The Decomposition of Coefficients of Power-series and the Divergence of Interpolation Polynomials*

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Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a function single valued and analytic within the circle $C_\rho: |z|=\rho>0$ but not analytic regular on C_ρ, and $S_n(z)$ be partial sums of the power series of respective degrees n, that is $S_n(z)=\sum_{k=0}^{n}a_kz^k$. Then it is known that the sequence of polynomials $S_n(z)$ of respective degrees n converges to $f(z)$ throughout the interior of the circle C_ρ, uniformly for any closed set interior to C_ρ, and diverges at every point exterior to C_ρ as n tends to infinity. And moreover, we have

$$\overline{\lim}_{n\to\infty} |S_n(z)|^{\frac{1}{n}}=\frac{|z|}{\rho} \qquad \text{for } z \text{ exterior to } C_\rho.$$

Above properties can be generalized to the sequence of polynomials found by interpolation to $f(z)$ in the points which satisfy a certain condition. (T. Kakehashi: *On the convergence-region of interpolation polynomials*, Journal of the Mathematical Society of Japan, 1955, Vol. 7).

In this paper, we consider the divergence property of the sequence found by interpolation in the set of points more generalized than that considered in the above paper.

Let the sequence of points

$$\text{(P)} \qquad \begin{cases} z_1^{(1)} \\ z_1^{(2)}, z_2^{(2)} \\ z_1^{(3)}, z_2^{(3)}, z_3^{(3)} \\ \cdots\cdots \\ z_1^{(n)}, z_2^{(n)}, z_3^{(n)}, \dots, z_n^{(n)} \\ \cdots\cdots \end{cases}$$

which do not lie exterior to the unit circle $C: |z|=1$, satisfy the condition that the sequence of

$$\frac{w_n(z)}{z^n}=\frac{(z-z_1^{(n)})(z-z_2^{(n)})\cdots(z-z_n^{(n)})}{z^n}$$

converges to a function $\lambda(z)$, single valued, analytic, and non-vanishing for z exterior to C, and uniformly for any finite closed points set exterior to C, that is

$$\text{(C)} \qquad \lim_{n\to\infty}\frac{w_n(z)}{z^n}=\lambda(z)\neq 0 \qquad \text{for } |z|>1.$$

Let $f(z)$ be a function single valued and analytic throughout the interior of the circle $C_\rho: |z|=\rho>1$ but not analytic regular on

C_ρ. Then the sequence of polynomials $P_n(z;f)$ of respective degrees n which interpolate to $f(z)$ in all the zeros of $w_{n+1}(z)$ is given by

$$\text{(I)}\qquad P_n(z;f)=\frac{1}{2\pi i}\int_{C_R}\frac{w_{n+1}(t)-w_{n+1}(z)}{w_{n+1}(t)}\frac{f(t)}{t-z}dt,\quad (1<R<\rho).$$

It is known that the sequence of polynomials $P_n(z;f)$ converges to $f(z)$ throughout the interior of the circle C_ρ as $n\to\infty$, and uniformly for any closed set interior to C_ρ. The divergence of the sequence $P_n(z;f)$ at every point exterior to C_ρ shall be ascertained in the following

Theorem. *Let the function $f(z)$ be single valued and analytic throughout the interior of the circle $C_\rho: |z|=\rho>1$, and (P) be the points set which satisfies the condition (C). Then the sequence of polynomials $P_n(z;f)$ of respective degrees n found by interpolation to $f(z)$ in all the zeros of $w_{n+1}(z)$ diverges at every point exterior to C_ρ. Moreover, we have*

$$(1)\qquad \overline{\lim}_{n\to\infty}|P_n(z;f)|^{\frac{1}{n}}=\frac{|z|}{\rho}\qquad \text{for } |z|>\rho>1.$$

In the proof of this theorem, it is convenient to have several lemmas.

Lemma 1. *Let $\{A_n\}: n=0,1,2,\dots$ be a sequence of complex numbers which satisfy*

$$(2)\qquad \overline{\lim}_{n\to\infty}|A_n|^{\frac{1}{n}}=1.$$

Then there exists a sequence of positive numbers $\lambda_n: n=0,1,2,\dots$ which satisfies following conditions, that is

$$(3)\qquad 1=\lambda_0\leqq\lambda_1\leqq\lambda_2\leqq\cdots \quad \text{or}\quad 1=\lambda_0\geqq\lambda_1\geqq\lambda_2\geqq\cdots>0 \ (\textit{monotone}),$$

$$(4)\qquad \lim\frac{\lambda_{n+1}}{\lambda_n}=1,$$

and

$$(5)\qquad \overline{\lim}_{n\to\infty}|c_n|\equiv\overline{\lim}_{n\to\infty}\left|\frac{A_n}{\lambda_n}\right|=1.$$

In the case when $\infty>\overline{\lim}_{n\to\infty}|A_n|=A>0$, it is clear that the sequence $\lambda_0=1$, $\lambda_1=\lambda_2=\cdots=A$ satisfies above conditions.

In the case when $\overline{\lim}|A_n|=\infty$, we can choose the greatest positive integer α_1 such that $|A_{\alpha_1}|^{\frac{1}{\alpha_1}}=\max|A_n|^{\frac{1}{n}}>1$ because $\overline{\lim}|A_n|^{\frac{1}{n}}=1$. Next we can take the greatest positive integer $\alpha_2(>\alpha_1)$ such that $\left|\frac{A_{\alpha_2}}{A_{\alpha_1}}\right|^{\frac{1}{\alpha_2-\alpha_1}}=\max_{n>\alpha_1}\left|\frac{A_n}{A_{\alpha_1}}\right|^{\frac{1}{n-\alpha_1}}>1$ which is smaller than $|A_{\alpha_1}|^{\frac{1}{\alpha_1}}$. In the sequel, we can choose the infinite sequence of positive integers $\alpha_1<\alpha_2<\alpha_3<\cdots$ such that α_k is the greatest positive integer which satisfies

$$\left|\frac{A_{\alpha_k}}{A_{\alpha_{k-1}}}\right|^{\frac{1}{\alpha_k-\alpha_{k-1}}}=\max_{n>\alpha_{k-1}}\left|\frac{A_n}{A_{\alpha_{k-1}}}\right|^{\frac{1}{n-\alpha_{k-1}}}>1,$$

and

$$|A_{\alpha_1}|^{\frac{1}{\alpha_1}}>\left|\frac{A_{\alpha_2}}{A_{\alpha_1}}\right|^{\frac{1}{\alpha_2-\alpha_1}}>\left|\frac{A_{\alpha_3}}{A_{\alpha_2}}\right|^{\frac{1}{\alpha_3-\alpha_2}}>\cdots$$

which must converge to 1 by the condition $\overline{\lim}\,|A_n|^{\frac{1}{n}}=1$.

If we arrange the sequence of positive numbers

$$1=\lambda_0<\lambda_1<\lambda_2<\cdots$$

by

$$\lambda_1=|A_{\alpha_1}|^{\frac{1}{\alpha_1}},\ \lambda_2=|A_{\alpha_1}|^{\frac{2}{\alpha_1}},\ldots,\ \lambda_{\alpha_1}=|A_{\alpha_1}|,$$

$$\lambda_{\alpha_1+1}=|A_{\alpha_1}|\left|\frac{A_{\alpha_2}}{A_{\alpha_1}}\right|^{\frac{1}{\alpha_2-\alpha_1}},\ \lambda_{\alpha_1+2}=|A_{\alpha_1}|\left|\frac{A_{\alpha_2}}{A_{\alpha_1}}\right|^{\frac{2}{\alpha_2-\alpha_1}},\ldots,\ \lambda_{\alpha_2}=|A_{\alpha_2}|,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\lambda_{\alpha_k+1}=|A_{\alpha_k}|\left|\frac{A_{\alpha_{k+1}}}{A_{\alpha_k}}\right|^{\frac{1}{\alpha_{k+1}-\alpha_k}},\ \lambda_{\alpha_k+2}=|A_{\alpha_k}|\left|\frac{A_{\alpha_{k+1}}}{A_{\alpha_k}}\right|^{\frac{2}{\alpha_{k+1}-\alpha_k}},\ldots,\ \lambda_{\alpha_{k+1}}=|A_{\alpha_{k+1}}|,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

we have, for any positive integer n, putting $n=\alpha_k+\nu$ for $\alpha_k<n<\alpha_{k+1}$

$$\left|\frac{A_n}{\lambda_n}\right|=\left|\frac{A_{\alpha_k+\nu}}{\lambda_{\alpha_k+\nu}}\right|=\left\{\left|\frac{A_{\alpha_k+\nu}}{A_{\alpha_k}}\right|^{\frac{1}{\nu}}\left|\frac{A_{\alpha_{k+1}}}{A_{\alpha_k}}\right|^{\frac{-1}{\alpha_{k+1}-\alpha_k}}\right\}^{\nu}\leqq 1$$

or

$$\left|\frac{A_n}{\lambda_n}\right|=\left|\frac{A_{\alpha_k}}{\lambda_{\alpha_k}}\right|=1\qquad\text{for } n=\alpha_k,\ k=1,2,\ldots,$$

and

$$\frac{\lambda_{n+1}}{\lambda_n}=\frac{\lambda_{\alpha_k+\nu+1}}{\lambda_{\alpha_k+\nu}}=\left|\frac{A_{\alpha_{k+1}}}{A_{\alpha_k}}\right|^{\frac{1}{\alpha_{k+1}-\alpha_k}}:\quad \nu=0,1,2,\ldots,(\alpha_{k+1}-\alpha_k-1),\ k=1,2,\ldots$$

which converges to 1 as $n\to\infty$.

Hence we have the monotone increasing sequence of positive numbers λ_n which satisfies relations (3), (4), and (5). Thus the lemma has been proved for the case $\overline{\lim}\,|A_n|=\infty$.

In the case when $\lim A_n=0$, we can choose a positive integer α_1 such that $1>|A_{\alpha_1}|>|A_n|$ for $n>\alpha_1$, and take the greatest positive integer α_2 such that $|A_{\alpha_2}|=\max_{\alpha_1<n}|A_n|<|A_{\alpha_1}|$. In such a way, we can choose a sequence of positive integers $\alpha_1<\alpha_2<\alpha_3<\cdots$ such that $|A_{\alpha_k}|>|A_{\alpha_{k+1}}|>|A_n|$ for $n>\alpha_{k+1}$, and $|A_n|\leqq|A_{\alpha_{k+1}}|<1$ for $\alpha_k<n<\alpha_{k+1}$. It is clear that the sequence $\{A_{\alpha_k}\}: k=1,2,\ldots$ satisfies the relation $\overline{\lim}_{k\to\infty}|A_{\alpha_k}|^{\frac{1}{\alpha_k}}=1$.

Next we take the positive integer β_1 equal to α_1, and we can choose the smallest positive number $\beta_2(>\beta_1)$ from $\{\alpha_k\}$ such that

$1>\left|\frac{A_{\beta_2}}{A_{\beta_1}}\right|^{\frac{1}{\beta_2-\beta_1}}>|A_{\beta_1}|^{\frac{1}{\beta_1}}$. In such a way, we can choose the sequence of positive numbers $\beta_1<\beta_2<\beta_3<\cdots$ from the sequence $\{\alpha_k\}$, such that β_{k+1} is the smallest positive integer which satisfies

$$1>\left|\frac{A_{\beta_{k+1}}}{A_{\beta_k}}\right|^{\frac{1}{\beta_{k+1}-\beta_k}}>\left|\frac{A_{\beta_k}}{A_{\beta_{k-1}}}\right|^{\frac{1}{\beta_k-\beta_{k-1}}}: k=2,3,\dots .$$

It is clear that the sequence $\left\{\left|\frac{A_{\beta_{k+1}}}{A_{\beta_k}}\right|^{\frac{1}{\beta_{k+1}-\beta_k}}\right\}$ converges to 1 as $k\to\infty$.

If we arrange the sequence of positive numbers

$1=\lambda_0>\lambda_1>\lambda_2>\cdots$

by

$$\lambda_1=|A_{\beta_1}|^{\frac{1}{\beta_1}},\ \lambda_2=|A_{\beta_1}|^{\frac{2}{\beta_1}},\dots,\ \lambda_{\beta_1}=|A_{\beta_1}|,$$

$$\lambda_{\beta_1+1}=|A_{\beta_1}|\left|\frac{A_{\beta_2}}{A_{\beta_1}}\right|^{\frac{1}{\beta_2-\beta_1}},\ \lambda_{\beta_1+2}=|A_{\beta_1}|\left|\frac{A_{\beta_2}}{A_{\beta_1}}\right|^{\frac{2}{\beta_2-\beta_1}},\dots,\ \lambda_{\beta_2}=|A_{\beta_2}|,$$

$$\dots\dots\dots\dots\dots\dots$$

$$\lambda_{\beta_k+1}=|A_{\beta_k}|\left|\frac{A_{\beta_{k+1}}}{A_{\beta_k}}\right|^{\frac{1}{\beta_{k+1}-\beta_k}},\ \lambda_{\beta_k+2}=|A_{\beta_k}|\left|\frac{A_{\beta_{k+1}}}{A_{\beta_k}}\right|^{\frac{2}{\beta_{k+1}-\beta_k}},\dots,\ \lambda_{\beta_k}=|A_{\beta_k}|,$$

$$\dots\dots\dots\dots\dots\dots,$$

we have, for any positive integer $n(>\beta_1)$, putting $n=\beta_k+\nu$ for $\beta_k<n<\beta_{k+1}$,

$$\left|\frac{A_n}{\lambda_n}\right|=\left|\frac{A_{\beta_k+\nu}}{\lambda_{\beta_k+\nu}}\right|=\left\{\left|\frac{A_{\beta_k+\nu}}{A_{\beta_k}}\right|^{\frac{1}{\nu}}\left|\frac{A_{\beta_{k+1}}}{A_{\beta_k}}\right|^{\frac{-1}{\beta_{k+1}-\beta_k}}\right\}^{\nu}\leqq 1$$

or

$$\left|\frac{A_n}{\lambda_n}\right|=\left|\frac{A_{\beta_k}}{\lambda_{\beta_k}}\right|=1 \qquad \text{for } n=\beta_k: k=1,2,3,\dots,$$

and

$$\lim_{n\to\infty}\frac{\lambda_{n+1}}{\lambda_n}=1.$$

Thus the sequence $\{\lambda_n\}: n=0,1,2,\dots$ satisfies (3), (4), and (5). Hence the lemma is valid also for the case $\lim A_n=0$. The lemma is thus established.

By this lemma, we can verify that the function $f(z)$ single valued and analytic within the circle $C_\rho: |z|=\rho$, but not analytic regular on C_ρ, can be expanded into the power series

$$(6)\qquad f(z)=\sum_{n=0}^{\infty}A_n\left(\frac{z}{\rho}\right)^n=\sum_{n=0}^{\infty}c_n\lambda_n\left(\frac{z}{\rho}\right)^n,$$

where λ_n and c_n satisfy the conditions (3), (4), and (5).

Lemma 2. *Let* $f(z)=\sum_{n=0}^{\infty}c_n\lambda_n\left(\frac{z}{\rho}\right)^n$ *be the function which satisfies*

(3), (4), and (5), and $\varphi(z)$ be a function single valued, analytic, and non-vanishing on $C_\rho : |z|=\rho>0$. If we put

$$f(z)\varphi(z)\equiv\sum_{n=-\infty}^{\infty}\gamma_n\left(\frac{z}{\rho}\right)^n, \tag{7}$$

we have

$$\infty>\overline{\lim}_{n\to\infty}\frac{|\gamma_n|}{\lambda_n}>0. \tag{8}$$

If we put $\varphi(z)\equiv\sum_{n=-\infty}^{\infty}\alpha_n\left(\frac{z}{\rho}\right)^n$, it is clear by the condition of the lemma that $\overline{\lim}_{p\to\infty}|\alpha_{-p}|^{\frac{1}{p}}=r<1$. And by the relation $\gamma_n=\sum_{p=0}^{\infty}c_p\lambda_p\alpha_{n-p}$ and $\overline{\lim}_{p\to\infty}\left|\frac{c_p\lambda_p}{\lambda_n}\right|^{\frac{1}{p-n}}=1$, for a positive integer n, we have

$$\frac{|\gamma_n|}{\lambda_n}\leqq\frac{|\sum_{p=0}^{\infty}c_p\lambda_p\alpha_{n-p}|}{\lambda_n}\leqq M\sum_{p=0}^{\infty}(1+\delta)^{|n-p|}(r+\delta)^{n-p}$$
$$\leqq 2M\sum_{p=0}^{\infty}(1+\delta)^p(r+\delta)^p$$

for any positive number δ, where M is a positive number independent of n and p. The last side is convergent for δ sufficiently small by the condition of $\varphi(z)$, and hence we can verify that $\frac{|\gamma_n|}{\lambda_n}<\infty$ uniformly for any positive integer n. Thus we can verify that $\overline{\lim}_{n\to\infty}\frac{|\gamma_n|}{\lambda_n}<\infty$. By this method, we can also verify that $|\gamma_{-n}|<\infty$ for any positive integer n.

Next we shall prove the relation $\overline{\lim}\frac{|\gamma_n|}{\lambda_n}>0$. If we put $\frac{1}{\varphi(z)}\equiv\sum_{n=-\infty}^{\infty}\beta_n\left(\frac{z}{\rho}\right)^n$ which is single valued and analytic on C_ρ, we have $\overline{\lim}_{n\to\infty}|\beta_n|^{\frac{1}{n}}<1$ and $\overline{\lim}_{n\to\infty}|\beta_{-n}|^{\frac{1}{n}}<1$. From the relation $f(z)=\sum_{n=0}^{\infty}c_n\lambda_n\left(\frac{z}{\rho}\right)^n=\sum_{n=0}^{\infty}\left(\sum_{p=-\infty}^{\infty}\gamma_{n-p}\beta_p\right)\left(\frac{z}{\rho}\right)^n$, we have

$$c_n=\frac{1}{\lambda_n}\sum_{p=-\infty}^{\infty}\gamma_{n-p}\beta_p=\sum_{p=-\infty}^{\infty}\frac{\lambda_{n-p}}{\lambda_n}\frac{\gamma_{n-p}}{\lambda_{n-p}}\beta_p \tag{9}$$

where $\lambda_k=1$ for $k<0$.

If we assume $\lim_{n\to\infty}\frac{\gamma_n}{\lambda_n}=0$, $\max_{-\infty<n<\infty}\left|\frac{\gamma_n}{\lambda_n}\right|\equiv M$ exists for any integer (positive or negative) n. For any two integers n and p, and for any $\delta>0$, we can verify by the condition of $\{\lambda_n\}: n=0, \pm1, \pm2, \ldots$ that there exists a positive number A, independent of n and p, which satisfies

$$\frac{\lambda_{n-p}}{\lambda_n}\leqq A(1+\delta)^{|p|}. \tag{10}$$

Accordingly, we have

$$|c_n|\leqq\max_{p\leqq 0}\frac{|\gamma_{n-p}|}{\lambda_{n-p}}\sum_{p=-\infty}^{0}\frac{\lambda_{n-p}}{\lambda_n}|\beta_p|+\max_{1\leqq p\leqq m}\frac{|\gamma_{n-p}|}{\lambda_{n-p}}\sum_{p=1}^{m}\frac{\lambda_{n-p}}{\lambda_n}|\beta_n|$$

$$+M\sum_{p=m+1}^{\infty}\frac{\lambda_{n-p}}{\lambda_n}|\beta_p| \leqq A\max_{q\geq 0}\frac{|\gamma_{n+q}|}{\lambda_{n+q}}\sum_{q=0}^{\infty}|\beta_{-q}|(1+\delta)^q$$
$$+A\max_{1\leq p\leq m}\frac{|\gamma_{n-p}|}{\lambda_{n-p}}\sum_{p=1}^{m}(1+\delta)^p|\beta_p|+AM\sum_{p=m+1}^{\infty}|\beta_p|(1+\delta)^p.$$

Here we can choose $\delta>0$ such that $\overline{\lim}_{n\to\infty}|\beta_{\pm n}|^{\frac{1}{n}}<\frac{1}{1+\delta}$. For any $\varepsilon>0$, if we take m sufficiently large, the last term becomes less than $\frac{\varepsilon}{3}$. And for a fixed number m, if we take n sufficiently large, the first and second terms become respectively less than $\frac{\varepsilon}{3}$. Hence we have $\lim_{n\to\infty}c_n=0$ which contradicts the assumption $\overline{\lim}_{n\to\infty}|c_n|=1$. Thus the lemma is established.

Lemma 3. *Let* $f(z)\equiv\sum_{n=0}^{\infty}c_n\lambda_n\left(\frac{z}{\rho}\right)^n$ *be the function which satisfies (3), (4), and (5), and* $\{\varphi_n(z)\}: n=1, 2,\dots$ *be a sequence of functions single valued and analytic on a closed domain, which contains the circle* C_ρ *in its interior, such that*

(11) $\lim_{n\to\infty}\varphi_n(z)=0,$ *uniformly on the domain.*

If we put

(12) $$f(z)\varphi_n(z)=\sum_{k=-\infty}^{\infty}\gamma_k^{(n)}\left(\frac{z}{\rho}\right)^k.$$

Then we have

(13) $$\lim_{n\to\infty}\frac{\gamma_n^{(n)}}{\lambda_n}=0.$$

If we put $\varphi_n(z)=\sum_{k=-\infty}^{\infty}\alpha_k^{(n)}\left(\frac{z}{\rho}\right)^k$, we can choose a positive number δ_0 such that $\varphi_n(z)$ are single valued and analytic on and between the two circles $C_{(1+\delta_0)\rho}$ and $C_{(1+\delta_0)^{-1}\rho}$, and we have

$$\alpha_k^{(n)}=\frac{\rho^k}{2\pi i}\int_{C_{(1+\delta_0)\rho}}\varphi_n(t)t^{-k-1}dt: k=0, 1, 2,\dots,$$
$$\alpha_k^{(n)}=\frac{\rho^k}{2\pi i}\int_{C_{(1+\delta_0)^{-1}\rho}}\varphi_n(t)t^{-k-1}dt: k=-1, -2,\dots .$$

Accordingly, we can verify that for any integer k

(14) $$|\alpha_k^{(n)}|\leqq M_n(1+\delta_0)^{-|k|},$$

where M_n can be allowed to approach zero as $n\to\infty$.

From the equation $\gamma_n^{(n)}=\sum_{p=0}^{\infty}c_p\lambda_p\alpha_{n-p}^{(n)}$, we have

$$\frac{\gamma_n^{(n)}}{\lambda_n}=\sum_{p=0}^{\infty}c_p\frac{\lambda_p}{\lambda_n}\alpha_{n-p}^{(n)}=\sum_{q=-\infty}^{\infty}c_{n-q}\frac{\lambda_{n-q}}{\lambda_n}\alpha_q^{(n)}$$
$$=\sum_{q=0}^{\infty}c_{n+q}\frac{\lambda_{n+q}}{\lambda_n}\alpha_{-q}^{(n)}+\sum_{q=1}^{\infty}c_{n-q}\frac{\lambda_{n-q}}{\lambda_n}\alpha_q^{(n)}.$$

For any positive number δ less than δ_0, if we put $K\equiv\max|c_n|$, we have

$$\frac{|\gamma_n^{(n)}|}{\lambda_n} \leqq KAM_n \sum_{q=0}^{\infty}\left(\frac{1+\delta}{1+\delta_0}\right)^q + KAM_n \sum_{q=1}^{\infty}\left(\frac{1+\delta}{1+\delta_0}\right)^q$$

by (10) and (14). By $M_n \to 0$, $\frac{\gamma_n^{(n)}}{\lambda_n}$ clearly tends to zero as $n \to \infty$. Thus the lemma is established.

The following lemma follows at once from lemmas 2 and 3.

Lemma 4. *Let* $f(z)=\sum_{n=0}^{\infty} c_n \lambda_n \left(\frac{z}{\rho}\right)^n$ *be the function which satisfies* (3), (4), *and* (5), *and* $\{\varphi_n(z)\}: n=1,2,\ldots$ *be the sequence of functions single valued and analytic on* C_ρ *such that*

(15) $\lim_{n\to\infty} \varphi_n(z)=\Phi(z)$ *(non-vanishing on* C_ρ*)*

uniformly on a closed domain which contains the circle C_ρ *in its interior.*

If we put $f(z)\varphi_n(z)=\sum_{k=-\infty}^{\infty}\gamma_k^{(n)}\left(\frac{z}{\rho}\right)^k$, *then we have*

(16) $$\infty > \overline{\lim}_{n\to\infty} \frac{|\gamma_n^{(n)}|}{\lambda_n} > 0.$$

Now we are in a position to prove the theorem. For the function $f(z)=\sum_{n=0}^{\infty} c_n\lambda_n\left(\frac{z}{\rho}\right)^n : \rho>1$ which satisfy (3), (4), and (5), and for any point z exterior to C_ρ, the sequence of functions

$$\varphi_n(t) \equiv \frac{w_{n+1}(t)-w_{n+1}(z)}{w_{n+1}(t)(t-z)}\left(\frac{t}{z}\right)^{n+1} = \frac{t^{n+1}}{w_{n+1}(t)}\left(\frac{w_{n+1}(t)}{z^{n+1}} - \frac{w_{n+1}(z)}{z^{n+1}}\right)\frac{1}{t-z};$$
$$n=0,1,2,\ldots$$

converges to $\frac{-\lambda(z)}{\lambda(t)(t-z)}$ (non-vanishing on C_ρ) uniformly on the closed domain $1<R'' \leqq |t| \leqq R' < |z|$, $(R''<\rho<R')$. By lemma 4 and the equality (I), we can verify that

$$\overline{\lim}_{n\to\infty} \frac{1}{\lambda_n}\left|\left(\frac{\rho}{z}\right)^{n+1} P_n(z;f)\right|$$
$$= \overline{\lim}_{n\to\infty} \frac{1}{\lambda_n}\left|\left(\frac{\rho}{z}\right)^{n+1} \frac{1}{2\pi i}\int_{C_R} \frac{w_{n+1}(t)-w_{n+1}(z)}{w_{n+1}(t)} \frac{f(t)}{t-z}dt\right|$$
$$= \overline{\lim}_{n\to\infty} \frac{\rho}{\lambda_n}\left|\frac{\rho^n}{2\pi i}\int_{C_R} \varphi_n(t) f(t)^{-n-1} dt\right| = \overline{\lim}_{n\to\infty}\frac{\rho}{\lambda_n}|\gamma_n^{(n)}|$$
for $|z|>\rho$

is bounded and positive. Now the relation

$$\overline{\lim}_{n\to\infty} |P_n(z;f)|^{\frac{1}{n}} = \frac{|z|}{\rho} \qquad \text{for } |z|>\rho$$

follows at once by $\lim_{n\to\infty} \sqrt[n]{\lambda_n}=1$ which can be verified by (4), and hence $P_n(z;f)$ can not converge for every z exterior to C_ρ. Thus the theorem is established.