119. Generalization of the Concept of Cohomology of Finite Groups

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The aim of the present note is to sketch foundations to establish relations between the cohomology theory and the theory of representations of finite groups. It is obtained through certain generalization of the concept of cohomology groups. From the thus generalized standpoint the ordinary cohomology theory of finite groups is seen as a local theory at a point in a certain space. Besides the interest of the thus obtained so to say global cohomology theory itself, this generalization is effective in applications of the cohomology theory. The author will discuss it before long in other chance.

1. Let G be a finite group, A be an abelien group such that G induces some automorphisms in A as a right operator group, i.e. for each s in G

$$a \to a^s \qquad (a \in A,\ s \in G)$$

is an automorphism and

$$(a^s)^t = a^{st} \qquad (t \in G).$$

Let Z be the ring of rational integers, $||$ be a normalized valuation of the rational number field R. We denote by $Z_{||}$ the ring Z of rational integers itself or the ring of l-adic integers, according to each of cases when $||$ is the normalized archimedien valuation $||_\infty$ or when $||$ is a normalized non archimedien valuation $||_l$ corresponded to a prime natural number l, respectively. Let D be a representation of G with regular matrices with coefficients in $Z_{||}$. We call such a pair of a normalized valuation of R and a representation of G in $Z_{||}$ as a point in the space of cohomology of G. From now on we define the local cohomology group of G with A as coefficients at a point in the space of cohomology of G as follows.

2. Let $A^{||i}$ for $i=1,2,3,\dots$ be the trivial subgroup in A consisting only of the unit element e or the subgroup consisting of every l^i-th power of elements in A, according to each of cases when $||=||_\infty$ or when $||=||_l$, respectively. Let $A_i^{(||)}$ denote the quotient group $A/A^{||i}$. As $A^{||i}$ is G-subgroup, $A_i^{(||)}$ is a G-right group. Let $\bar{A}^{(||)}$ be the inverse limit group

$$\bar{A}^{(||)} = \varprojlim [A_i^{(||)};\ L_i^{i+j}] \quad (i=1,2,\dots;\ j=0,1,\dots)$$

where we denote by L_i^{i+j} the natural homomorphism of $A_{i+j}^{(||)}$ onto

$A_i^{(||)}$. As L_i^{i+j} is clearly G-homomorphism, $\bar{A}^{(||)}$ is G-right group. We denote by $(a)_i$ for $a \in \bar{A}^{(||)}$ the $A_i^{(||)}$-component of a. We define the operation of $Z_{||}$ on $\bar{A}^{(||)}$ as follows. When $||$ is $||_\infty$, it is clear. When $||$ is non archimedien, i.e. $||=||_l$ with a prime natural number l, we write each element ν in $Z_{||}$ in the normal form as

$$\nu=\nu_0+\nu_1 l+\nu_2 l^2+\cdots$$

where ν_j ($j=0,1,2,\dots$) are non negative rational integers not greater than $l-1$. We denote by $[\nu]_j$ the partial sum

$$[\nu]_j=\nu_0+\nu_1 l+\cdots+\nu_j l^j.$$

As the normal form of elements is $Z_{||}$ is uniquely determined by ν, $[\nu]_j$ is determined uniquely by ν for each $j=0,1,2,\dots$. From now on throughout this article we denote A as a G-right module, (i.e. as an additive group with G as right operator group). As

$$L_i^{i+j}([\nu]_{i+j-1}(a)_{i+j})=[\nu]_{i-1}(a)_i$$

for each $i=1,2,\dots$; $j=0,1,\dots$, $\varprojlim\,[[\nu]_{i-1}(a)_i;\ L_i^{i+j}]$ is an inverse spectrum in $\bar{A}^{(||)}$. We define the operation of $Z_{||}$ on $\bar{A}^{(||)}$ as

$$\nu a=a\nu=\varprojlim\,[[\nu]_{i-1}(a)_i;\ L_i^{i+j}]$$

for each $\nu \in Z_{||}$, a $\bar{A}^{(||)}$. Thus $\bar{A}^{(||)}$ is a $Z_{||}$-left and $G_{||}$-right module, where we denote by $G_{||}$ the group ring $Z_{||}[G]$ of G over $Z_{||}$.

3. Let d denote the degree of the representation D, and $Z_{||d}$ and $G_{||d}$ be the full matrix ring of degree d over $Z_{||}$ and the group ring $Z_{||d}[G]$ of G over $Z_{||d}$, respectively. Let $\bar{A}^{(||)}_d$ be the set of all matrices of degree d with coefficients in $\bar{A}^{(||)}$, and we make it $Z_{||d}$-left and $G_{||d}$-right module in the natural way. Let M be an arbitrary $G_{||d}$-right module. Let σ_D denote the element

$$\sigma_D=\textstyle\sum_{s\in D} sD(s)$$

in $G_{||d}$. Let $M(G, D)$ be the subgroup in M, not necessarily $Z_{||d}$-subgroup, consisting of all such elements m in M which satisfy

$$ms=mD(s^{-1})$$

for each s in G. As is easily seen, it holds

$$M(G, D)\supset M\sigma_D.$$

$M\sigma_D$ is a subgroup, not necessarily $Z_{||d}$-subgroup. We denote by $N(M, G, D)$ the quotient additive group

$$N(M, G, D)=M(G, D)/M\sigma_D.$$

4. Let σ_0 denote the element

$$\sigma_0=\textstyle\sum_{s\in G} s$$

in $G_{||}$. We denote by J the $G_{||}$-right module

$$J=G_{||}/\sigma_0 Z_{||}.$$

Let I_D denote the $G_{||d}$-right module consisting of all such elements m in $G_{||d}$ which satisfy

$$\textstyle\sum_{s\in G} D(s^{-1})\nu(s)=0,$$

where

$$m=\sum_{s\in G}s\nu(s) \qquad (\nu(s)Z_{||a})$$

and 0 denotes the zero matrix of degree d. We denote by J_p with rational integer $p\geqq 0$ the tensor product of p copies of J over $Z_{||}$ and make it $Z_{||}$-left and $G_{||}$-right module, defining the operation as

$$u\otimes u'\otimes\cdots\otimes u''\cdot s=us\otimes u's\otimes\cdots\otimes u''s$$

$$\nu\cdot u\otimes u'\otimes\cdots\otimes u''=u\otimes u'\otimes\cdots\otimes u''\cdot\nu=u\nu\otimes u'\otimes\cdots\otimes u''$$

with $u\otimes u'\otimes\cdots\otimes u''\in J_p$, $s\in G$, and $\nu\in Z$. Let $\mathfrak{O}$ be an associative, not necessarily commutative ring with unit element ε. Let M and N be both $\mathfrak{O}$-left and right modules with ε as trivial operator. We define the tensor product $M\underset{\mathfrak{O}}{\otimes}N$ of M and N over $\mathfrak{O}$ as the quotient group of the free additive group generated by the product set $M\times N$ of M and N by the subgroup generated by such elements as $m+m'\times n-m\times n-m'\times n$, $m\times n+n'-m\times n-m\times n'$, and $mr\times n-m\times rn$ $(r\in\mathfrak{O})$, and make it as $\mathfrak{O}$-left and right module, defining the operation as

$$r\cdot m\underset{\mathfrak{O}}{\otimes}n=rm\underset{\mathfrak{O}}{\otimes}n,\quad m\underset{\mathfrak{O}}{\otimes}n\cdot r=m\underset{\mathfrak{O}}{\otimes}nr$$

for each $r\in\mathfrak{O}$. The tensor product is associative, but not necessarily commutative. Let $[G_{||a}]_q$ be the tensor product of q copies of $G_{||a}$ over $Z_{||a}$, where q is rational integer $\geqq 0$. The subset in $[G_{||a}]_q$ consisting of classes involving elements in the product set $I_D\times I_D\times\cdots\times I_D$ of q copies of I_D is clearly $Z_{||a}$-right submodule in $[G_{||a}]_q$, which we denote by $[I_D]_q$, calling as tensor product of q copies of I_D in $G_{||a}$ over $Z_{||a}$. We make it $G_{||a}$-right module, defining the operation as

$$v\underset{[d]}{\otimes}v'\underset{[d]}{\otimes}\cdots\underset{[d]}{\otimes}v''\cdot s=vs\underset{[d]}{\otimes}v's\underset{[d]}{\otimes}\cdots\underset{[d]}{\otimes}v''s$$

$$v\underset{[d]}{\otimes}v'\underset{[d]}{\otimes}\cdots\underset{[d]}{\otimes}v''\cdot\nu=v\underset{[d]}{\otimes}v'\underset{[d]}{\otimes}\cdots\underset{[d]}{\otimes}v''\nu$$

for each $s\in G$ and $\nu\in Z_{||a}$. We denote by $[I_D]_q\underset{[d]}{\otimes}\bar{A}^{(||)}{}_a$ the tensor product over $Z_{||a}$ of $[I_D]_q$ as $Z_{||a}$-right module and $\bar{A}^{(||)}{}_a$ as $Z_{||a}$-left module and make it $G_{||a}$-right module, defining the operation as

$$v\underset{[d]}{\otimes}a\cdot s=vs\underset{[d]}{\otimes}as,\quad v\underset{[d]}{\otimes}a\cdot\nu=v\underset{[d]}{\otimes}a\nu$$

for each $s\in G$ and $\nu\in Z_{||a}$. Let $J_p\otimes[I_D]_q\underset{[d]}{\otimes}\bar{A}^{(||)}{}_a$ be the tensor product of J_p and $[I_D]_q\underset{[d]}{\otimes}\bar{A}^{(||)}{}_a$ over $Z_{||}$ and make it $G_{||a}$-right module, defining the operation as

$$u\otimes v\underset{[d]}{\otimes}a\cdot s=us\otimes vs\underset{[d]}{\otimes}as,\quad u\otimes v\underset{[d]}{\otimes}a\cdot\nu=u\otimes(v\underset{[d]}{\otimes}a)\nu$$

for each $s\in G$ and $\nu\in Z_{||a}$.

5. Now we define the local p-q-cohomology group $pqH(A, G, ||, D)$ at the point $(||, D)$ in the space of cohomology of G as

$$pqH(A, G, ||, D) = N(J_p \otimes [I_D]_q \underset{[d]}{\otimes} \bar{A}^{(||)}{}_d, G, D).$$

The thus defined local cohomology group at the point $(||_\infty, x_0)$ of the archimedien valuation $||_\infty$ and the trivial character x_0 coincides with Chevalley's definition of the usual cohomology group in [1]. It holds satisfactory analogue of fundamental theorems as to diagrams of exact sequences in the usual theory for our local cohomology groups at each point in the space of cohomology of G. So, our naming is not unreasonable, though it is not yet obtained any unicity theorem as to such generalization. We define the global cohomology groups as the direct sum of all local ones. The global concept in the cohomology theory will have perhaps new effects in applications, which the author will discuss some other time.

Added in proof. The author was remarked by Serre that $pqH(A, G, ||, D) \cong pqH(A', G, ||, x_0) = pqH(A', G)$, where $A' = J_p \otimes I_{x_0 q} \otimes \bar{A}^{(||)}d \underset{[d]}{\otimes} Z_d$ and G operates as $a'^s = a^s \underset{[d]}{\otimes} D(s^{-1})\, \nu$ for $a' = a \underset{[d]}{\otimes} \nu$, $a \in J_p \otimes I_{x_0 q} \otimes \bar{A}^{(||)}d$, $\nu \in Z_d$, and $s \in G$. So the concept of a local cohomology group in our sense adds nothing new to the cohomology theory itself. But it does not mean objections to the possibility of usefulness of such a representation of certain cohomology groups with reference to representations of G. The author thanks Serre and Chevalley for their kind remarks and suggestions.

Reference

[1] C. Chevalley: Class field theory, Nagoya (1954).