# 146. *Note on the Lebesgue Property in Uniform Spaces*

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The main purpose of this note is to characterize a separated uniform space having the Lebesgue property.

Following Prof. Kiyoshi Iséki [2], we say that an open covering[1] $\mathfrak{S}$ of a separated uniform space[2] $E$ has the *Lebesgue property* if there exists a surrounding[3] $V$ of $E$ such that, for each point $x \in E$, we can find a member $U$ of the covering $\mathfrak{S}$ containing the set $V(x)$.

In a previous paper [3], one of the authors studied mainly separated uniform spaces in which every open covering has the Lebesgue property and called such a uniform space to have the *Lebesgue property* (further a separated uniform space $E$ was referred to have the *finite Lebesgue property* if every finite open covering of $E$ has the Lebesgue property). However, it is natural to understand that they are properties of the uniform structure. In this point of view, these concepts should be defined for the uniform structure, but we adopt in this note the terminology used in [3] to avoid confusion.

THEOREM 1. *Let $E$ be a uniformisable Hausdorff topological space, then $E$ possesses a uniform structure compatible with the topology of $E$ for which $E$ has the Lebesgue property, if and only if $E$ is paracompact.*[4] *Under this condition, this uniform structure is the finest of all uniform structures compatible with the topology of $E$, and is unique.*

Proof. In [3], we obtained that every open covering of a separated uniform space has a star refinement[4] if the space has the Lebesgue property, so that the only if part of the theorem is obvious.

To prove the if part, let us consider the family $\mathfrak{F}$ of all open coverings of $E$. Since every open covering of $E$ has a star refinement, $\mathfrak{F}$ is a uniformity[5] as can be easily seen. It is not hard

1) A collection $\mathfrak{S}$ of open sets of a topological space $S$ is called an *open covering* of $S$, if the union of all members of $\mathfrak{S}$ is $S$. An open covering of $S$ is said to be *finite* if it consists of a finite number of open sets.

2) Cf. N. Bourbaki [1].

3) "entourage" in French.

4) Let $\mathfrak{S}$ be an open covering of a topological space $S$; an open covering $\mathfrak{S}'$ of $S$ is a *star refinement* of $\mathfrak{S}$ if, for any $V \in \mathfrak{S}'$, the union of all $U \in \mathfrak{S}'$ meeting $V$ is contained in some member of $\mathfrak{S}$, or *$\Delta$-refinement* of $\mathfrak{S}$ if, for any point $x \in S$, the union of all $U \in \mathfrak{S}'$ to which $x$ belongs is contained in some member of $\mathfrak{S}$. A Hausdorff topological space $S$ is *paracompact* if every open covering has a $\Delta$-refinement.

5) Cf. J. W. Tukey [5].

also to see that the uniformity $\mathfrak{F}$ is compatible with the topology of $E$. In fact, for any point $a \in E$ and any open set $U$ containing $a$, the complement of $a$ and $U$ make up an open covering of $E$, and so we can find its star refinement $\mathfrak{S}$ in $\mathfrak{F}$. But then the union of members of $\mathfrak{S}$ to which $a$ belongs can not be contained in the complement of $a$, and so in $U$. Now since every open covering has a star refinement, $E$ has the Lebesgue property for the uniform structure corresponding to the uniformity $\mathfrak{F}$. In view of the construction of $\mathfrak{F}$, this uniform structure is the finest of all compatible uniform structures, and the uniqueness follows from a theorem of A. Weil [6, p. 16].

As immediate consequences of the theorem, we have

COROLLARY 1. *A separated uniform space $E$ has the Lebesgue property if and only if $E$ is paracompact and every continuous mapping of $E$ into another uniform space is uniformly continuous.*

COROLLARY 2. *A separated uniform space $E$ has the Lebesgue property if and only if $E$ is paracompact and the uniform structure of $E$ is the finest of all uniform structures compatible with the topology of $E$.*

We showed in [3] that every uniform space having the Lebesgue property is complete, and hence we obtain

COROLLARY 3. *If a separated uniform space $E$ with the finest uniform structure is paracompact, then $E$ is complete.*

COROLLARY 4. *In a separated uniform space with the finest uniform structure, two concepts, paracompactness and the Lebesgue property coincide.*

COROLLARY 5. *Every compact space has the Lebesgue property.*

THEOREM 2. *Let $E$ be a uniformisable Hausdorff topological space, then $E$ possesses a uniform structure compatible with the topology of $E$ for which $E$ has the finite Lebesgue property, if and only if $E$ is normal.*

Proof. Since the only if part follows immediately from Corollary 1 of Theorem 3 in [3], we have only to prove the if part. Let $\mathfrak{F}$ be the family of all finite open coverings of $E$, then it is easy to see that $\mathfrak{F}$ is a basis of a uniformity and that this uniformity is compatible with the topology of $E$. For the uniform structure corresponding to this uniformity, the uniform space $E$ has the finite Lebesgue property since every finite open covering has a finite $\Delta$-refinement.[4] This completes the proof.

As we have seen above, in a normal space, the family of all finite open coverings form a basis of a uniformity compatible with the topology. We shall call the uniform structure corresponding to this uniformity the *F-structure*. It is clear that a normal space with the $F$-structure is precompact and that a uniform structure for which the space is precompact is less fine than the $F$-structure.

THEOREM 3. *In a normal space $E$, every compatible uniform structure for which $E$ has the finite Lebesgue property is finer than the $F$-structure. Conversely, for any compatible uniform structure finer than the $F$-structure, $E$ has the finite Lebesgue property.*

Proof. Let $\mathfrak{F}$ be a uniform structure compatible with the topology of $E$ for which $E$ has the finite Lebesgue property. Then for any finite open covering of $E$, we can take a surrounding $V$ ensured by the finite Lebesgue property. Therefore, the uniform structure $\mathfrak{F}$ must be finer than the $F$-structure. The latter part of the theorem is quite obvious.

COROLLARY 1. *A separated uniform space is precompact and has the finite Lebesgue property at the same time if and only if the uniform structure is equivalent to the $F$-structure.*

COROLLARY 2. *In a paracompact space $E$, every uniform structure for which $E$ has the finite Lebesgue property ensures the Lebesgue property if and only if $E$ is compact.*

COROLLARY 3. *A paracompact space with a unique uniform structure is compact.*

COROLLARY 4. *A paracompact uniform space $E$ with the finite Lebesgue property has the Lebesgue property if and only if the uniform structure is the finest of all uniform structures compatible with the topology of $E$.*

Accordingly, a result of A. A. Monteiro and M. M. Peixoto [4, Lemme 3] can be written as follows: *if a metric space has the finite Lebesgue property, its structure is the finest of all uniform structures compatible with the topology of the space.* Therefore, in view of Corollary 1 of Theorem 6 in [3], we have the following

COROLLARY 5. *In a normal space $E$, if a compatible uniform structure which is finer than the $F$-structure is metrisable, then $E$ can be decomposed into the union of a compact subset and a discrete subset.*

## References

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