144. On the Unstable Homotopy Groups of Spheres

By Tsuneyo YAMANOSHITA

(Comm. by Z. SUETUNA, M.J.A., Nov. 12, 1955)

Recently H. Toda[1] has determined the homotopy groups of spheres $\pi_{n+r}(S^n)$ $(r \leqq 13)$. The object of this note is to report on the results on the same subject which the author has obtained through computation on the cohomology algebras $H^*(S^n, n+i; Z_p)$. Our method allows as to verify Toda's results on the structure of these groups except for a point which will be indicated at the end of the note.

Denote by (X, i) a topological space such that $\pi_j(X, i)=\pi_j(X)$ for $j \geqq i$ and $\pi_j(X, i)=0$ for $j<i$. Then we have the Hurewicz isomorphism

$$H_i(X, i; Z)=\pi_i(X, i)=\pi_i(X) \qquad (i \geqq 2).$$

The p-component of $H_i(X, i; Z)$, and therefore that of $\pi_i(X)$ are determined by the cohomology groups $H^i(X, i; Z_p)$, $H^{i+1}(X, i; Z_p)$ and the cohomological relations between them.

For example let $X=S^3$. In the fibering $S^3/(S^3, 4)=K(Z, 3)$, the mod 2 cohomology algebra $H^*(Z, 3; Z_2)$ of the Eilenberg-MacLane space $K(Z, 3)$ is known to be the polynomial algebra $P[u_3, Sq^{(2^i,\dots,2)}u_3$ $(i=1, 2, \dots)]$,[2] and the consideration of the spectral sequence associated with the above fibering gives

$$H^*(S^3, 4; Z_2)=P[s]\otimes\wedge(Sq^1 s),$$

the transgression image of s being $\tau s=Sq^2 u_3$ $(\dim s=4)$.

Considering the fibering $(S^3, 3+i)/(S^3, 3+i+1)=K(\pi_{3+i}(S^3), 3+i)$ successively for $i=0, 1, \dots,$ we obtain the following table of 2-components of the homotopy groups of 3-sphere:

$$\begin{array}{llll}
(\pi_3(S^3)=Z), & {}^2\pi_4(S^3)=Z_2, & {}^2\pi_5(S^3)=Z_2, & {}^2\pi_6(S^3)=Z_4, \\
{}^2\pi_7(S^3)=Z_2, & {}^2\pi_8(S^3)=Z_2, & {}^2\pi_9(S^3)=0, & {}^2\pi_{10}(S^3)=0, \\
{}^2\pi_{11}(S^3)=Z_2, & {}^2\pi_{12}(S^3)=Z_2+Z_2, & {}^2\pi_{13}(S^3)=Z_4+Z_2, & \\
{}^2\pi_{14}(S^3)=Z_4+Z_2+Z_2, & {}^2\pi_{15}(S^3)=Z_2+Z_2, & {}^2\pi_{16}(S^3)=Z_2, & \\
{}^2\pi_{17}(S^3)=Z_2, & {}^2\pi_{18}(S^3)=Z_2, & {}^2\pi_{19}(S^3)=Z_2+Z_2, & \\
{}^2\pi_{20}(S^3)=Z_4+Z_2+Z_2, & {}^2\pi_{21}(S^3)=Z_4+Z_2+Z_2. & &
\end{array}$$

Similar procedure is applicable also for $p \neq 2$; in view of

1) H. Toda: Sur les groupes d'homotopie des sphères, Calcul de groupes d'homotopie des sphères, Compt. Rend., **240**, 42–44, 147–149 (1955).

2) J. P. Serre: Cohomologie mod 2 des complexes d'Eilenberg-MacLane, Comm. Math. Helv., **27**, 198–232 (1953).

$$H^*(Z, 3; Z_p) = \wedge(v_3, \mathfrak{P}^{p^i}\dots\mathfrak{P}^1 v_3 (i=0, 1, \dots)) \otimes P[\varDelta_p^1 \mathfrak{P}^{p^i}\mathfrak{P}^{p^{i-1}}\dots\mathfrak{P}^1 v_3 (i=0, 1, \dots)],^{3)}$$

we obtain

$$H^*(S^3, 4; Z_p) = P[t] \otimes \wedge(\varDelta_p^1 t),$$

where dim $t=2p$ and $\tau t = \mathfrak{P}^1 v_3$. (Therefore, the p-component of $\pi_{2p}(S^3)$ is Z_p $(p \geqq 2)$.) Here $\varDelta_p^1$ is the coboundary operator associated with the exact sequence

$$0 \longrightarrow Z_p \longrightarrow Z_{p^2} \longrightarrow Z_p \longrightarrow 0.$$

By repeated application of our method we can obtain the following table of 3-components ${}^3\pi_{3+i}(S^3)$ at least for $i \leqq 18$:

${}^3\pi_j(S^3) = 0$ except for the following cases $(j \leqq 21)$.

$${}^3\pi_6(S^3) = Z_3, \quad {}^3\pi_9(S^3) = Z_3, \quad {}^3\pi_{10}(S^3) = Z_3,$$
$${}^3\pi_{13}(S^3) = Z_3, \quad {}^3\pi_{14}(S^3) = Z_3, \quad {}^3\pi_{16}(S^3) = Z_3,$$
$${}^3\pi_{17}(S^3) = Z_3, \quad {}^3\pi_{18}(S^3) = Z_3, \quad {}^3\pi_{19}(S^3) = Z_3,$$
$${}^3\pi_{20}(S^3) = Z_3, \quad {}^3\pi_{21}(S^3) = Z_3.$$

Of course we can easily verify Moore's theorem[4] on the p-components of $\pi_j(S^3)$.

Furthermore by using theorems on the Freudenthal suspension, we can verify the results of H. Toda on the group structure of the homotopy groups of spheres. We have obtained a different result from Toda's on ${}^2\pi_{22}(S^9)$, namely ${}^2\pi_{22}(S^9) = Z_2$. The result announced by Toda seems to be a misprint. In all other cases, we have obtained the same results as Toda.

Our method is applicable also for calculation of homotopy groups of Lie groups and some homogeneous spaces.

3) H. Cartan: Sur les groupes d'Eilenberg-MacLane, II, Proc. Nat. Acad. Sci. U. S. A., **40**, 704–707 (1954).

4) J. C. Moore: On the homotopy groups of spaces with a single non-vanishing homology group, Ann. Math., **59**, 549–557 (1954).