154. On Semi-reducible Measures

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In the present note we consider the generalizations of Katětov's results concerning semi-reducibility of finite Baire measure in topological spaces.[1] Hereafter by a measure μ we mean a finite measure. We shall use the following notations: $C(X,R)$=all of real-valued continuous functions on a T-space X, $\mathfrak{B}^*(X)$=Baire family on X, $P(f)=\{x \mid f(x)>0,\ f\in C(X,R)\}$, $\mathfrak{P}(X)=\{P(f)\mid f\in C(X,R)\}$, $Z(f)=\{x\mid f(x)=0,\ f\in C(X,R)\}$, $\mathfrak{Z}(X)=\{Z(f)\mid f\in C(X,R)\}$, $Q(\mu)=\{x\mid \mu(U_x)>0$ for any neighborhood $U_x\in\mathfrak{B}^*(X)$ of $x\}$.[2]

Theorem 1. *Let X be a normal space. Then in order that every Baire measure in X be semi-reducible, it is necessary and sufficient that every Baire measure in any closed subspace is semi-reducible.*

Proof. Necessity has been proved by Katětov [2]. Hence we shall prove only sufficiency. Now suppose that there exists a Baire measure μ which is not semi-reducible. Then there exists a proper closed set $F\in\mathfrak{Z}(X)$ such that $\mu(F)>0$, $F\frown Q(\mu)=\phi$, even if $Q(\mu)$ is a null set. We restrict μ on $\mathfrak{B}^*(F)$ and represent it as μ_F. Then, by the hypothesis, there exists a closed set Q_1 in F, which semi-reduces μ_F. Suppose that Q_1 is not a null set and take a point $p\in Q_1\subset F$. Then $\mu(G)>0$ holds for every open set $G\in\mathfrak{P}(X)$ containing p; for $0<\mu_F(G\frown F)=\mu(G\frown F)\leqq\mu(G)$. This contradicts to the fact that $p\notin Q(\mu)$, and hence Q_1 is a null set. Therefore we have $\mu_F(F)=\mu(F)=0$. This is a contradiction and hence the proof is completed.

In the same way as in Theorem 1, we have the following result concerning two-valued measures.

Theorem 2. *Let X be a completely regular space. Then in order that every two-valued Baire measure in X be semi-reducible, it is necessary and sufficient that every two-valued measure in any proper closed subspace is semi-reducible.*

Remark. Since a completely regular space such that every two-valued Baire measure in X is semi-reducible is equivalent to a Q-space,[3] we have the same result as [6, Theorem 5].

1) See Katětov [2]. A measure μ defined on a σ-field $\mathfrak{B}$ is called semi-reducible if there exists a closed subset Q of X such that (1) $\mu(G)>0$ holds if G is open, $G\in\mathfrak{B}$, $G\frown Q\neq\phi$, and (2) $\mu(F)=0$ holds if F is closed, $F\in\mathfrak{B}$, $F\frown Q=\phi$.

2) In a completely regular space the set $Q(\mu)$ is obviously closed.

3) See Hewitt [1, Theorem 16].

Necessary conditions in Theorems 1 and 2 are strengthend as follows.

Theorem 3. (1) *Let X be a normal space such that every Baire measure in X is semi-reducible. Then every Baire measure in any F_σ-subspace $Y \subset X$ is also semi-reducible.*

(2) *Let X be a comletely regular space such that every two-valued Baire measure is semi-reducible. Then every two-valued Baire measure in any F_σ-subspace of X is also semi-reducible.*

Corollary. *Any F_σ-subspace of a Q-space X is also a Q-space.*

We state another proof of [2, Theorem 1] in a slightly different form.

Theorem 4. *Let X be a paracompact space. Then the following conditions are equivalent;*

(1) *every (two-valued) Baire measure in X is semi-reducible,*

(2) *every (two-valued) Borel measure in any closed discrete subspace of X is reducible,*

(3) *for any (two-valued) Baire measure μ, the union of a discrete collection of open sets $\{G_\alpha \mid G_\alpha \in \mathfrak{P}(X), \alpha \in A\}$ with μ-measure zero has also μ-measure zero.*[4]

Proof. (1)$\to$(2). This is obvious by Theorem 1. (2)$\to$(3). Let μ be a (two-valued) Baire measure and let $\{G_\alpha \mid \alpha \in A\}$ be a discrete collection of open sets such that $G_\alpha \in \mathfrak{P}(X)$ for any $\alpha \in A$ and that $\mu(G_\alpha)=0$. Setting $G=\bigcup_{\alpha\in A} G_\alpha$, we have $G \in \mathfrak{P}(X)$. Take a point $p_\alpha \in G_\alpha$ for any $\alpha \in A$ and let $Y=\{p_\alpha\}$. Then it is plain that Y is a closed discrete subspace and that $\bigcup_\alpha\{G_\alpha \mid p_\alpha \in E\} \in \mathfrak{P}(X)$ for any subset E of Y. Let $\nu(E)=\mu[\bigcup_\alpha G_\alpha \mid p_\alpha \in E]$. Then $\nu(E)$ is a (two-valued) Borel measure in Y and vanishes at each point p_α. Hence we obtain $\nu(Y)=\mu(G)=0$, from discreteness of Y and semi-reducibility of ν. (3)$\to$(1). Let μ be any (two-valued) Baire measure in X. We can suppose that $\mu(X)>0$, for, if $\mu(X)=0$, μ is semi-reducible. Take a closed set $F \in \mathfrak{B}^*(X)$ such that $F \frown Q(\mu)=\phi$. It is sufficient to see that $\mu(F)=0$. Since there exists a neighborhood $U_p \in \mathfrak{B}^*(X)$ of p such that $\mu(U_p)=0$ for any point $p \in F$, $\mathfrak{U}=\{U_p, F^c \mid p \in F\}$ is an open covering of X. Let $\mathfrak{V}=\{V_{n\alpha} \mid \alpha \in A_n, n=1,2,\dots\}$ be a locally finite refinement of $\mathfrak{U}$ such that $\{V_{n\alpha} \mid \alpha \in A_n\}$ is a discrete collection for each n and any $V_{n\alpha} \in \mathfrak{P}(X)$.[5] Now setting $G_n=\bigcup_\alpha\{V_{n\alpha} \mid V_{n\alpha} \frown F \neq \phi\}$, we have $\mu(G_n)=0$, by $\mu(V_{n\alpha})=0$ for any $V_{n\alpha} \subset G_n$ and (2). Therefore, setting $G=\bigcup_{n,\alpha}\{V_{n\alpha} \mid V_{n\alpha} \frown F \neq \phi\}$, it holds that $G \in \mathfrak{P}(X)$, $F \subset G$, and

4) A collection $F=\{H_\alpha \mid \alpha \in A\}$ of subsets of a T-space X is called discrete if (1) the closures $\overline{H}_\alpha$ are mutually disjoint, (2) $\bigcup_{\beta\in B}\overline{H}_\beta$ is closed for any subset B of A.

5) This is stated in Stone [7, Remark of Theorem 1].

$\mu(G)=0$, for $G=\bigcup_{n=1}^{\infty}G_n$. Hence we have $\mu(F)=0$ and this completes the proof.

Remark 1. The conditions (2) and (3) are replaced with the following (2′) and (3′) respectively.

(2′) Any closed discrete subset has the power of (two-valued) measure zero.

(3′) Any discrete collection of open subsets has the power of (two-valued) measure zero.[6]

Remark 2. It should be noticed that in the proof of Theorem 4 the fact that a paracompact space is strong screenable plays an important rôle.[7]

In the following we state some results concerning the Lindelöf property.

Theorem 5. *Let X be a normal space. If for any Baire measure μ in X there exists a closed subspace S with the Lindelöf property such that $\mu(F)=0$ holds for any closed set $F\in\mathfrak{B}^*(X)$ with $F\frown S=\phi$, then any closed discrete subset of X has the power of measure zero.*

Proof. Let $Y=\{p_\alpha \mid \alpha\in A\}\subset X$ be a closed discrte subset and let ν be a Borel measure defined on all subsets of Y such that $\nu(p_\alpha)=0$ for $\alpha\in A$. We define a Baire measure μ in X as follows:

$$\mu(B)=\nu(B\frown Y) \quad \text{for any Baire set } B\subset X.$$

The set $Y\frown S$ being countable, we have $\mu(Y\frown S)=0$. On the other hand the set $Y-Y\frown S$ being closed and disjoint with S, there exists a closed set $Z\in\mathfrak{Z}(X)$ such that $Z\frown S=\phi$, $Y-Y\frown S\subset Z$. Then we have $\mu(Z)=0$ by the hypothesis, which shows that $\nu(Y-Y\frown S)=0$. Hence we have $\nu(Y)=0$. This completes the proof.

Remark. If we replace the word "Baire" with "Borel", the result above is valid in a T_1-space, except the closedness of S.

Theorem 6. *Let X be a paracompact space. If any closed discrete subset in X has the power of measure zero, then for any Baire measure μ in X there exists a closed subspace S with the Lindelöf property such that S semi-reduces μ.*

Proof. Since for any Baire measure μ there is a closed subset S such that S semi-reduces μ by Theorem 4, it is sufficient to see that the set S has the Lindelöf property. The subspace S being paracompact as a closed subspace of X, any open covering $\mathfrak{U}=\{G_\alpha \mid \alpha\in A\}$ of S has a locally finite refinement $\mathfrak{B}=\{H_{n\alpha} \mid \alpha\in A, n=1,2,\dots\}$ of $\mathfrak{U}$ such that $\{H_{n\alpha} \mid \alpha\in A\}$ $(n=1,2,\dots)$ is a discrete collection of

6) About the power of (two-valued) measure zero, see Marczewski and Sikorski [3].

7) See Michael [4] and Nagami [5].

open subsets in S. Put $H_{n\alpha}=S\frown U_{n\alpha}$, where $U_{n\alpha}$ is open in X. Let $\{F_{n\alpha}\mid\alpha\in A, n=1,2,\dots\}$ be a closed covering of S such that $F_{n\alpha}\subset H_{n\alpha}$, $F_{n\alpha}\neq\phi$. Then we can take discrete collections of open subsets $\{V_{n\alpha}\mid\alpha\in A_n\}$ $(n=1,2,\dots)$ in X such that $F_{n\alpha}\subset V_{n\alpha}\subset U_{n\alpha}, V_{n\alpha}\in\mathfrak{P}(X)$. Since $V_{n\alpha}\frown S\neq\phi$, we have $\mu(V_{n\alpha})>0$, and hence A_n is a countable set for each n. This shows that $\mathfrak{U}$ is a countable open covering of S. This completes the proof.

By Theorems 5 and 6, we have the following

Theorem 7. *Let X be a paracompact space. Then the following two conditions are equivalent;*

(1) *for any Baire measure μ in X, there exists a closed subspace S with the Lindelöf property such that $\mu(F)=0$ holds for any closed set $F\in\mathfrak{B}^*(X)$ with $F\frown S=\phi$,*

(2) *any closed discrete subset of X has the power of measure zero.*

Theorem 8. *Let X be a paracompact and perfectly normal space. Then the following two conditions are equivalent;*

(1) *for any Borel measure μ in X there exists a decomposition of X such that $X=N+S$, where $\mu(N)=0$ and S has the Lindelöf property,*

(2) *any closed discrete subset of X has the power of measure zero.*

We note that the following theorem is necessary to prove the decomposition theorems in Marczewski and Sikorski [3] from our results.

Theorem 9. *Let X be a metric space. Then the following conditions are equivalent;*

(1) *the character*[8] *of X has measure zero,*

(2) *the power of any closed discrete subset has measure zero,*

(3) *the power of any discrete open collection has measure zero.*

Proof. (1)→(2). This follows from [3, Theorem 2]. It is trivial that (2) and (3) are equivalent. (3)→(1). Let k be any positive integer and let $U_{1/k}(p)$ be a sphere with a centre p and radius $1/k$; then an open covering $\mathfrak{U}_k=\{U_{1/k}(p)\mid p\in X\}$ has a locally finite refinement $\mathfrak{B}_k=\{V_{n\alpha}^{(k)}\mid\alpha\in A_n, n=1,2,\dots\}$ such that $\{V_{n\alpha}^{(k)}\}$ is a discrete collection for each n and k and that $V_{n\alpha}^{(k)}\in\mathfrak{P}(X)$, for a metric space is also paracompact. Then obviously all of open sets belonging to $\mathfrak{B}_k(k=1,2,\dots)$ constitute an open base with the power of measure zero.

8) By the character of a T-space we mean the smallest power of an open basis of X.

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