# 7. Notes on Topological Spaces. I. A Theorem on Uniform Spaces

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The object of this Note is to give a formulation of a theorem on metric space. We suppose that *all spaces considered are separated.*

Let $X$ be a uniform space. The space $X$ is called an *absolute closed* if whenever $X$ is topologically imbedded in a uniform space $Y$, then $X$ is a closed in $Y$.

We can easily obtain the following

*Theorem 1. A uniform space is absolutely closed if and only if it is complete in sense of uniform structure.*

Proof. Let $X$ be an absolutely closed uniform space, and $\hat{X}$ the completion of $X$. Then $X\subset\hat{X}$ and $X$ is closed in $\hat{X}$. Therefore, by a well-known proposition (N. Bourbaki [1], p. 149, Prop. 6, or G. Nöbeling [2], p. 200, 26.4), $X$ is complete.

Conversely, let $X$ be a complete uniform space, and suppose that $X$ is imbedded in a uniform space $Y$. Since the same proposition 6 of N. Bourbaki stated in the first part of proof, and $Y$ is separated, $X$ is closed. Therefore $X$ is absolutely closed.

From Theorem 1, we can reduce the following interesting

*Theorem 2. Let $X$ be a uniform space and $Z$ any uniform space containing $X$. If there is a complete uniform space $Y$ containing $X$ and $X$ is a $G_\delta$-set in $Y$, then $X$ is the intersection of a closed set and a $G_\delta$-set of $Z$.*

Proof. Let $Y$ be a complete uniform space satisfying the condition of Theorem 2. Let $Z$ be any uniform space containing $X$. Since $X$ is a $G_\delta$-set in $Y$, there are countable closed sets $F_n$ of $Y$ such that $X=Y-\bigcup_{n=1}^{\infty}F_n$. By Theorem 1, $Y$ is absolutely closed. On the other hand, if we let $Y\cup Z$ be a uniform space, $Y$ is closed in $Y\cup Z$. Hence each closed set $F_n$ is closed in $Y\cup Z$ and therefore $F_n$ and $Y$ are closed in $Z$.

The identity

$$X=Y\frown\bigcap_{n=1}^{\infty}(Z-F_n)$$

implies that $X$ is the intersection of a closed set and a $G_\delta$-set of $Z$.

Q.E.D.

Conversely, we have easily seen the following

*Theorem 3. Let $X$ be a uniform space. If, for every uniform space $Z$ containing $X$, $X$ is the intersection of a closed set and a $G_\delta$-set in $Z$, there is a complete uniform spxce $Y$ containing $X$ and $X$ is a $G_\delta$-set in $Y$.*

Let $Y$ be the completion of $X$, then the uniform space $Y$ has the desired property.

*Definition 1.* A uniform space $X$ is said to be *uniformly complete* if there is a complete uniform space $Y$ such that $X$ is a $G_\delta$-set in $Y$.

Then we have the

*Theorem 4. A uniform space is uniformly complete if and only if it is a $G_\delta$-set in the completion.*

Proof. If a uniform space $X$ is a $G_\delta$-set in the completion $\hat{X}$, then $X$ is uniform complete, since $\hat{X}$ is complete. Let $X$ be a uniformly complete space. Then there is a complete space $Y$ such that $X$ is a $G_\delta$-set in $Y$. Let $\overline{X}$ be the closure of $X$ in $Y$. Then $\overline{X}$ is a complete space containing $X$ and $X$ is dense in $\overline{X}$ and a $G_\delta$-set in $\overline{X}$. Let $\hat{X}$ be the completion of $X$. Consider the identity mapping $i$ on $X$, then $i$ is a uniformly continuous mapping from $X$ to $\overline{X}$. Since $X$ is dense in $\hat{X}$, the identity mapping $i$ can be continuously extended on $\hat{X}$ to $\overline{X}$.

Hence, $X$ is a $G_\delta$-set in $\hat{X}$, since $X$ is a $G_\delta$-set in $\overline{X}$. This completes the proof.

Therefore, by Theorem 2, we have the following

*Theorem 5. A uniform space $X$ is uniformly complete, if and only if $X$ is the intersection of a closed set and a $G_\delta$-set in every uniform space containing $X$.*

## References

[1] N. Bourbaki: Topologie Générale, Chaps. 1–2, Hermann, Paris.

[2] G. Nöbeling: Grundlagen der analytischen Topologie, Berlin-Göttingen-Heidelberg, Springer (1954).