3. *Closed Mappings and Metric Spaces*

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A mapping of a topological space X onto another topological space Y is said to be closed if the image of every closed subset of X is closed in Y. Concerning the problem: "Under what condition is the image of a metric space under a closed continuous mapping metrizable?", several interesting results have been obtained recently by G. T. Whyburn [6], A. V. Martin [3], and V. K. Balachandran [1]. In the present note, we shall give an answer to this problem by proving that the image space Y of a metric space X under a closed continuous mapping f is metrizable if and only if the boundary $\mathfrak{B}f^{-1}(y)$ of the inverse image $f^{-1}(y)$ is compact for every point y of Y. A problem raised by Balachandran [1] will also be solved.

1. We shall prove

Lemma 1. *Let f be a closed continuous mapping of a normal T_1-space X onto a topological space Y. If Y satisfies the first countability axiom, then $\mathfrak{B}f^{-1}(y)$ is countably compact for every point y of Y.*

Proof. Let y be any point of Y. By the first countability axiom, there exists a countable collection $\{V_i \mid i=1,2,\cdots\}$ of open neighborhoods of y such that for any open neighborhood U there can be found some V_i with $V_i \subset U$.

Suppose that $\mathfrak{B}f^{-1}(y)$ is not countably compact. Then there exist a countable number of points x_i, $i=1,2,\cdots$ of $\mathfrak{B}f^{-1}(y)$ such that $\{x_i\}$ has no limit point. Then by the normality of X we can find a discrete collection $\{G_n\}$ of open sets of X such that

$$x_i \in G_i \quad \text{for } i=1,2,\cdots; \; G_i \cap G_j = 0 \quad \text{for } i \neq j$$

and $\{G_n\}$ is locally finite.

Since each point x_i belongs to the boundary $\mathfrak{B}f^{-1}(y)$ of $f^{-1}(y)$, there exists a point x_i' of X such that

$$x_i' \notin f^{-1}(y), \quad x_i' \in G_i \cap f^{-1}(V_i).$$

Then $\{x_i' \mid i=1,2,\cdots\}$ is locally finite in X and hence the set C consisting of all points x_i', $i=1,2,\cdots$ is closed. Therefore if we put $H=Y-f(C)$, H is an open set of Y. Since $x_i' \notin f^{-1}(y)$, we have $y \in H$. Hence there exists some V_i such that $V_i \subset H$. This implies that we have $f(x_i') \notin V_i$ for some i. On the other hand we have chosen the point x_i' so that $x_i' \in f^{-1}(V_i)$. This is a contradiction. Thus Lemma 1 is proved.

2. We shall now establish the following theorem.

Theorem 1. *Let f be a closed continuous mapping of a metric space X onto a topological space Y. In order that Y be metrizable it is necessary and sufficient that the boundary $\mathfrak{B}f^{-1}(y)$ of the inverse image $f^{-1}(y)$ be compact for every point y of Y.*

Proof. Since the necessity is an immediate consequence of Lemma 1, we have only to prove the sufficiency.

(i) Let $\mathfrak{B}f^{-1}(y)$ be compact for each point y of Y. We shall define an open set $L(y)$ as follows:

$$L(y)=\begin{cases}\text{the interior of } f^{-1}(y), & \text{if } \mathfrak{B}f^{-1}(y)\neq 0,\\ f^{-1}(y)-p_y, & \text{if } \mathfrak{B}f^{-1}(y)=0,\end{cases}$$

where p_y is an arbitrary point of $f^{-1}(y)$. We put

$$X_0=X-L,\quad L=\cup\{L(y)\mid y\in Y\}.$$

Then X_0 is a closed subset of X. If we denote by φ the inclusion map of X_0 into X (that is, $\varphi(x)=x$ for $x\in X_0$), then $g=f\varphi$ is a closed continuous mapping of X_0 onto Y such that

$$g^{-1}(y)=\begin{cases}\mathfrak{B}f^{-1}(y), & \text{if } \mathfrak{B}f^{-1}(y)\neq 0,\\ p_y, & \text{if } \mathfrak{B}f^{-1}(y)=0.\end{cases}$$

Hence $g^{-1}(y)$ is compact for every point y of Y.

(ii) By (i) we may and shall assume that f is a closed continuous mapping of a metric space X onto Y such that $f^{-1}(y)$ is compact for every point y of Y. Y is clearly a T_1-space.

Let $\mathfrak{M}_i$ be a locally finite closed covering of X which consists of sets of diameter $<1/i$. Let us put $\mathfrak{N}_i=f(\mathfrak{M}_i)=\{f(M)\mid M\in\mathfrak{M}_i\}$. Since for each point y of Y $f^{-1}(y)$ is compact, there exists an open set G containing $f^{-1}(y)$ such that G intersects only finitely many elements of $\mathfrak{M}_i$. If we put $H=Y-f(X-G)$, we have $y\in H, f^{-1}(H)\subset G$, and hence H intersects only finitely many elements of $\mathfrak{N}_i$. This shows that $\mathfrak{N}_i$ is a locally finite closed covering of Y.

Let V be any open neighborhood of y. Then $f^{-1}(y)\subset f^{-1}(V)$. Since $f^{-1}(y)$ is compact, the distance between $f^{-1}(y)$ and $X-f^{-1}(V)$ is positive and hence we have $S(f^{-1}(y),\mathfrak{M}_i)\subset f^{-1}(V)$ for some i, where $S(A,\mathfrak{M}_i)$ means the union of the sets of $\mathfrak{M}_i$ which intersect A. Therefore we have $S(y,\mathfrak{N}_i)\subset V$ for some i.

In the previous paper [5] the following metrizability condition was obtained:

In order that a T_1-space be metrizable it is necessary and sufficient that there exists a countable collection $\{\mathfrak{K}_i\mid i=1,2,\cdots\}$ of locally finite closed coverings of the space such that for any neighborhood U of any point x there exists some $\mathfrak{K}_i$ satisfying the condition $S(x,\mathfrak{K}_i)\subset U$.

Therefore Y is metrizable. This completes our proof.

3. The following theorem in the previous paper [2] is a corollary to Theorem 1; indeed the part (ii) of the above proof of Theorem 1 is nothing but a proof of this theorem.

Theorem 2. *Let f be a closed continuous mapping of a metric space X onto a topological space Y. If the inverse image $f^{-1}(y)$ is compact for every point y of Y,*[1] *then Y is metrizable.*

We shall give another proof of this theorem by virtue of the following theorem.

Theorem 3. *Let f be a closed continuous mapping of a T_1-space X onto a topological space Y. Then if X is normal or collectionwise normal, so also is Y. Furthermore, in case $\mathfrak{B}f^{-1}(y)$ is compact for every point y of Y, if X is paracompact and normal, so also is Y.*

Proof. (i) Let $\{F_\alpha\}$ be a discrete collection of closed sets in Y. Then $\{f^{-1}(F_\alpha)\}$ is clearly a discrete collection of closed sets in X. Let X be collectionwise normal; then there exists a system of disjoint open sets G_α of X such that $f^{-1}(F_\alpha)\subset G_\alpha$ for each α. If we put $H_\alpha=Y-f(X-G_\alpha)$, we have $f^{-1}(F_\alpha)\subset f^{-1}(H_\alpha)\subset G_\alpha$ and hence $H_\alpha\cap H_\beta=0$ for $\alpha\neq\beta$. This proves that Y is collectionwise normal. The proof for the case of normality is now obvious from the above argument.

(ii) Let X be paracompact and normal, and let $\mathfrak{B}f^{-1}(y)$ be compact for every point y of Y. As in the proof of Theorem 1 we may assume that $f^{-1}(y)$ is compact for each point y.

Let $\mathfrak{G}$ be any open covering of Y. Then $\mathfrak{H}=\{f^{-1}(G)\mid G\in\mathfrak{G}\}$ is an open covering of X, and there exists a locally finite closed covering $\mathfrak{M}$ of X which is a refinement of $\mathfrak{H}$. If we put $\mathfrak{N}=\{f(M)\mid M\in\mathfrak{M}\}$, $\mathfrak{N}$ is a locally finite closed covering of Y and is a refinement of $\mathfrak{G}$, as is shown in (ii) of the proof of Theorem 1. Since Y is normal, by a theorem of E. Michael [4] Y is paracompact.

Proof of Theorem 2. Since X is paracompact and normal, Y is paracompact and normal by Theorem 3, so that Y is fully normal. Let $O^{(n)}(x)\,(n=1,2,\cdots)$ be an open sphere with the center x and the radius $1/n$ for each point x of X. For any point y of Y, we have $\bigcup_{x\in f^{-1}(y)} O^{(n)}(x)\supset f^{-1}(y)$. Let $G^{(n)}(y)=Y-f(X-\bigcup_{x\in f^{-1}(y)} O^{(n)}(x))$; then $G^{(n)}(y)$ is an open set containing y since f is closed and continuous. Let $\mathfrak{G}_n=\{G^{(n)}(y)\mid y\in Y\}\,(n=1,2,\cdots)$; then each $\mathfrak{G}_n$ is a normal covering of Y, since Y is fully normal. For the proof of the metrizability of Y, we have only to prove that for all points $y\in Y$, $\{S(y,\mathfrak{G}_n)\mid n=1,2,\cdots\}$ is a basis for neighborhoods of y.

Let y be any point of Y and U any open neighborhood of y.

1) In [2] this condition was left out in the statement of the theorem, and the proof there contains an error, but it can be corrected.

Then, since $f^{-1}(y)$ is compact, $\rho[f^{-1}(y), X-f^{-1}(U)]=d>0$ where ρ is the metric for X. Let m be a positive integer such that $1/m<d/2$ and let l be a positive integer such that $1/l<\rho[f^{-1}(y), X-f^{-1}(G^{(m)}(y))]$, $l>m$. Then $y \in G^{(l)}(y')$ implies $y' \in G^{(m)}(y)$.

In fact, suppose on the contrary that $y' \notin G^{(m)}(y)$; then $f^{-1}(y') \subset X-f^{-1}(G^{(m)}(y))$. Hence

$$(*) \qquad 1/l<\rho[f^{-1}(y), X-f^{-1}(G^{(m)}(y))] \leqq \rho[f^{-1}(y), f^{-1}(y')].$$

Since $y \in G^{(l)}(y')$, $f^{-1}(y) \subset f^{-1}(G^{(l)}(y')) \subset \bigcup_{x \in f^{-1}(y')} O^{(l)}(x)$. Hence for any point $x \in f^{-1}(y)$, there exists a point x' such that $x \in O^{(l)}(x')$ and $x' \in f^{-1}(y')$. Therefore $\rho[f^{-1}(y), f^{-1}(y')] \leqq 1/l$, contrary to $(*)$.

We shall prove that $S(y, \mathfrak{G}_l) \subset U$. Let y'' be any point of $S(y, \mathfrak{G}_l)$; then there exists $G^{(l)}(y')$ with some y' such that $y'' \in G^{(l)}(y')$ and $y \in G^{(l)}(y')$. Then $y' \in G^{(m)}(y)$. Hence for any point $x \in f^{-1}(y'')$, there exist points x'' and x''' such that $x \in O^{(l)}(x'')$, $x'' \in f^{-1}(y')$, $x'' \in O^{(m)}(x''')$ and $x''' \in f^{-1}(y)$. Then $\rho(x, x''') \leqq 1/l+1/m<2/m<d$. Hence $x \in f^{-1}(U)$, so that $y'' \in U$. Therefore $\{S(y, \mathfrak{G}_n) \mid n=1, 2, \cdots\}$ is a basis for neighborhoods of y. This completes our proof.

4. By combining Theorem 1 with Lemma 1 we obtain at once the following theorem, which was essentially proved by G. T. Whyburn [6] for the case where X is separable.

Theorem 4. *Let f be a closed continuous mapping of a metric space X onto a topological space Y. If Y satisfies the first countability axiom, then Y is metrizable.*

In case A is a closed subset of a metric space X, the space obtained from X by contracting A to a point is the image of X under the natural mapping which is a closed mapping. Therefore the image of a metric space under a closed continuous mapping is not always metrizable.

Thus a problem raised by Balachandran [1] is answered by Theorems 1 and 4.

5. As is easily shown (cf. [1, Lemma 1]), if a T_1-space X satisfies the first countability axiom, so also does the image of X under an open continuous mapping. Hence we obtain the following theorem of Balachandran [1] from Theorem 4.

Theorem 5. *The image of a metric space under any closed, open, continuous mapping is metrizable.*

6. On the basis of Theorem 1 we have

Theorem 6. *Let f be a closed continuous mapping of a metric space X onto another metric space Y. If X is separable or locally compact, so also is Y.*

Proof. By Theorem 1, $\mathfrak{B}f^{-1}(y)$ is compact for every point y of Y. By the closed continuous mapping g of X_0 onto Y which was defined in the part (i) of the proof of Theorem 1, our theorem for these two properties is reduced to theorems in the previous paper [2]; of course the proof in [2] can be modified easily so as to yield a direct proof for our case.

References

[1] V. K. Balachandran: A mapping theorem for metric spaces, Duke Math. Jour., **22**, 461–464 (1955).

[2] S. Hanai: On closed mappings, Proc. Japan Acad., **30**, 285–288 (1954).

[3] A. V. Martin: Decompositions and quasi-compact mappings, Duke Math. Jour., **21**, 463–469 (1954).

[4] E. Michael: A note on paracompact spaces, Bull. Amer. Math. Soc., **4**, 831–838 (1953).

[5] K. Morita: A condition for the metrizability of topological spaces and for n-dimensionality, Sci. Rep. Tokyo Kyoiku Daigaku, Section A, **5**, No. 114, 33–36 (1955).

[6] G. T. Whyburn: Open and closed mappings, Duke Math. Jour., **17**, 69–74 (1950).