1. *Evans's Theorem on Abstract Riemann Surfaces with Null-Boundaries. I*

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G. C. Evans[1] proved the following

Evans's theorem. *Let F be a closed set of capacity zero in the 3-dimensional euclidean space (or z-plane). Then there exists a positive unit-mass-distribution on F such that the potential engendered by this distribution has limit ∞ at every point of F.*

Let R^* be a null-boundary Riemann surface and let $\{R_n\}$ $(n=0, 1, 2, \cdots)$ be its exhaustion with compact relative boundaries $\{\partial R_n\}$. Put $R=R^*-R_0$. After R. S. Martin,[2] we introduce ideal boundary points as follows. Let $\{p_i\}$ be a sequence of points of R tending to the ideal boundary of R and let $\{G(z, p_i)\}$ be Green's function of R with pole at p_i. Let $\{G(z, p_{i_j})\}$ be a subsequence of $\{G(z, p_i)\}$ which converges to a function $G(z, p)$ uniformly in R. We say that $\{p_{i_j}\}$ determines a Martin's point p and we make $G(z, p)$ correspond to p. Furthermore Martin defined the distance between two points p_1 and p_2 of R or of the boundary by

$$\delta(p_1, p_2)=\sup_{z\in R_1-R_0}\left|\frac{G(z, p_1)}{1+G(z, p_1)}-\frac{G(z, p_2)}{1+G(z, p_2)}\right|.$$

It is clear that Martin's point p coincides with an ordinary point when $p\in R$ and that if $p_i\overset{\mathfrak{M}}{\rightarrow}p$,[3] $G(z, p_i)\rightarrow G(z, p)$ uniformly in R. In the following, we denote by $\overline{R}$[4] the sum of R and the set B of all ideal boundary points of Martin. Let p be a point of $\overline{R}$ and let $V_m(p)$ be the domain of R such that $\varepsilon[G(z, p)\geqq m]$. Then

Lemma 1. $$\int_{\partial V_m(p)}\frac{\partial G(z, p)}{\partial n}ds=2\pi:{}^{5)}\qquad m\geqq 0.$$

Proof. Let $p=\overset{\mathfrak{M}}{\lim_i} p_i$: $p\in B$, $p_i\in R$. Then $\underset{R-V_m(p_i)}{D}[G(z, p_i)]=2\pi m$ and

1) G. C. Evans: Potential and positively infinite singularities of harmonic functions, Monatschefte Math. U. Phys., **43** (1936).

2) R. S. Martin: Minimal positive harmonic functions, Trans. Amer. Math. Soc., **49** (1941).

3) In this paper $\mathfrak{M}$ means "with respect to Martin's metric".

4) The topology induced by this metric restricted in R is homeomorphic to the original topology and it is clear that B and $\overline{R}$ are closed and compact.

5) In this article, we denote by ∂A the relative boundary of A.

$$\underset{R-V_m(p)}{D}[G(z,p)] \leqq \underset{i}{\underline{\lim}} \underset{R-V_m(p_i)}{D}[G(z,p_i)] = 2\pi m.$$

Let $\omega_n(z)$ be a harmonic function in $R-R_0$ such that $\omega_n(z)=0$ on ∂R_0, $\omega_n(z)=M_n$ on ∂R_n and $\int_{\partial R_0} \frac{\partial \omega_n}{\partial n} ds = 2\pi$. Then since R is a null-boundary Riemann surface, $\lim_n M_n = \infty$. Put $z_n = e^{\omega_n + ih_n} = r_n e^{i\theta_n}$, where $h_n(z)$ is the conjugate of $\omega_n(z)$. Denote the curve on which $|z_n|=r$ by θ_n^r and the part of θ_n^r, contained in $R-V_m(p)$ by $\bar{\theta}_n^r$. Then $\int_{\bar{\theta}_n^r} d\theta_n \leqq 2\pi$.

$$\text{Put} \quad L(r_n) = \int_{\bar{\theta}_n^r} \left| \frac{\partial G(z,p)}{\partial r_n} \right| r_n \, d\theta_n. \quad \text{Then}$$

$$L^2(r_n) \leqq 2\pi r_n \int_{\bar{\theta}_n^r} \left| \frac{\partial G(z,p)}{\partial r_n} \right|^2 r_n \, d\theta_n,$$

$$\underset{R_n-V_m(p)}{D}[G(z,p)] = \int\int \left\{ \left(\frac{\partial G}{\partial r_n} \right)^2 + \frac{1}{r_n^2} \left(\frac{\partial G}{\partial \theta_n} \right)^2 \right\} r_n \, dr_n \, d\theta_n. \quad \text{Hence}$$

$$\int_1^{e^{M_n}} \frac{L^2(r_n)}{2\pi r_n} dr_n \leqq \int_1^{e^{M_n}} \frac{dD}{dr_n} dr_n \leqq 2\pi m, \text{ for every } n. \quad \text{Therefore}$$

there exists a sequence $\{L(r_n^i)\}$: $i=i(n)$ such that $L(r_n^i) \to 0$, when $n \to \infty$. $\int_{\partial R_0} \frac{\partial G(z,p)}{\partial n} ds = \int_{\bar{\theta}_n^r} \frac{\partial G(z,p)}{\partial n} ds + \int_{\partial \bar{V}_m(p)} \frac{G(z,p)}{\partial n} ds$, $\frac{\partial G(z,p)}{\partial n} \geqq 0$ on $\partial V_m(p)$; where $\bar{V}_m(p)$ is the part of $V_m(p)$ out of θ_n^r. Hence we have the lemma. When $p \in R$, our assertion is obvious.

Lemma 2. *Let $v_n(p)$ be an $\mathfrak{M}$-neighbourhood such that $v_n(p) = \underset{\bar{z} \in \bar{R}}{\varepsilon}\left[\delta(\bar{z}, p) < \frac{1}{n}\right]$. Then for every $V_m(p)$, there exists a neighbourhood $v_n(p)$ such that*

$$v_n(p) \subset V_m(p).$$

Proof. Assume that the lemma is false, there exists a sequence $\{q_i\}$ such that $\lim_i q_i = q^*$: $q_i \notin V_m(p)$ and $\delta(q^*, p)=0$. Let $\{G(z, q_i)\}$ be the corresponding functions to $\{q_i\}$. Take an ordinary neighbourhood $\mathfrak{B}(p)$[6] of p with a compact relative boundary such that

$$\int_{\partial V_l(p) \cap C\mathfrak{B}(p)} \frac{G(z,p)}{\partial n} ds \geqq \pi: \quad l \geqq 7m.$$

Since $q_i \notin V_m(p)$ and by the manner in Lemma 1 and by Green's formula, we have

$$m \geqq G(q_i, p) = \frac{1}{2\pi} \int_{\partial V_l(p)} G(z, q_i) \frac{\partial G(z,p)}{\partial n} ds.$$

6) $\mathfrak{B}(p)$ is such that $\mathfrak{B}(p) \ni R$ and $\mathfrak{B}(p) \cap R_{n_0} = 0$ for a certain n_0, we can choose as $\mathfrak{B}(p)$ one of component of $R - R_{n_0}$ containing p.

Therefore $$\frac{1}{2\pi}\int_{\partial V_i(p)\cap C\mathfrak{V}(p)} G(z, q_i)\frac{\partial G(z, p)}{\partial n} \leqq m,$$
accordingly there exist points $\{r_i\}$ on $\partial V_i(p)\cap C\mathfrak{V}(p)$ such that $G(r_i, q_i)\leqq 2m$, whence $G(r, q^*)\leqq 2m$: (r is one of limiting points of r_i) ($\in \partial V_i \cap C\mathfrak{V}$). Since $G(r, p)=l$, this fact means that $\delta(q^*, p)>\delta$ ($\delta>0$). Hence we have the lemma.

In the following, if $G(z, p)$ has a limit when $z\to q(\in B)$, we define the value of $G(z, p)$ at q by the above limit denoted by $G(q, p)$.

Lemma 3. *If at least one of p and q is contained in R,*
$$G(p, q)=G(q, p).$$
Assume $B \ni p$ ($p=\overset{\mathfrak{M}}{\lim}\, p_i$: $p_i \in R$) and $q \in R$. Let $\mathfrak{V}(p)$ be an ordinary neighbourhood of p with a compact relative boundary such that $\mathfrak{V}(p)\supset V_m(p)$ and $\mathfrak{V}(p) \not\ni q$. Then we have by Green's formula
$$\int_{\partial\mathfrak{V}(p)} G(z, p_i)\frac{\partial G(z, q)}{\partial n}ds-\int_{\partial\mathfrak{V}(p)} G(z, q)\frac{\partial G(z, p_i)}{\partial n}ds=G(p_i, q).$$
Since $p_i\overset{\mathfrak{M}}{\to}p$, $G(z, p_i)\to G(z, p)$ and $\frac{\partial G(z, p_i)}{\partial n}ds\to\frac{\partial G(z, p)}{\partial n}ds$ uniformly on $\partial\mathfrak{V}(p)$, each term of the left hand has its limit when $p_i\overset{\mathfrak{M}}{\to}p$, hence $G(p_i, q)$ has a limit $G(p, q)$. On the other hand $G(p_i, q)=G(q, p_i)$ and $G(q, p)=\lim_i G(q, p_i)$, hence $G(z, q)$ is $\mathfrak{M}$-continuous in $\bar{R}$ and $G(p, q)=G(q, p)$. $G(p, q)$ can be defined by another way as follows.

In the sequel, we suppose that both p and q lie on B and consider $G(z, q)$ in the neighbourhood of p. Let $V_m(p)=\underset{z}{\varepsilon}[G(z, p)\geqq m]$, $V_n(q)=\underset{z}{\varepsilon}[G(z, q)\geqq n]$ and put
$$\tilde{G}^M(z, q)=\min[M, G(z, q)].\quad\text{Then } \underset{R}{D}[\tilde{G}^M(z, q)]\leqq 2\pi M.$$
Let $G^M_{V_m}(z, q)$ be the lower envelope of non negative continuous superharmonic functions in $R-R_0$ which are larger than $\tilde{G}^M(z, q)$ in $R-R_0-V_m(p)$. Then $G^M_{V_m}(z, q)$ is harmonic in $V_m(p)$, continuous on $\partial V_m(p)\cap R$ and by Dirichlet principle $D_{V_m}[G^M_{V_m}(z, q)]\leqq D_{V_m}[\tilde{G}^M(z, p)]\leqq 2\pi M$.

Hence we can prove, by the same manner used in Lemma 1, that there exists a sequence of compact curves $\{C_i\}$ enclosing B such that $\{C_i\}$ tends to B when $i\to\infty$ and $\lim_{i=\infty}\int_{C_i\cap(V_m(p)-V_{m'}(p))}\left|\frac{\partial G(z, p)}{\partial n}\right|ds=0$ and we can prove that
$$\int_{\partial V_m(p)} G^M_{V_m}(z, q)\frac{\partial G(z, p)}{\partial n}ds=\int_{\partial V_{m'}(p)} G^M_{V_m}(z, q)\frac{\partial G(z, p)}{\partial n}ds, \tag{1}$$
where $m'>m$, i.e. $V_m(p)\supset V_{m'}(p)$.

Now let $G_{V_m}(z, q)$ be the lower envelope of non negative continuous superharmonic functions in $R-R_0$ which are larger than $G(z,q)$ in $R-R_0-V_m(p)$. Since $G^M_{V_m}(z,q)\uparrow G'_{V_m}(z,q)$ on $V_m(p)$ and the function $G(z)[=G(z,q)$, if $z\notin V_m(p)$, $=G'_{V_m}(z,q)$ on $V_m(p)]$ is one of superharmonic functions which are larger than $G(z,q)$ in $R-R_0-V_m(p)$, hence $G_{V_m}(z,q)=\lim_{M=\infty} G^M_{V_m}(z,q)$. Thus, let $M\to\infty$. Then by (1)

$$\int_{\partial V_m(p)} G(z,q)\frac{\partial G(z,p)}{\partial n}ds=\int_{\partial V_m(p)} G_{V_m}(z,q)\frac{\partial G(z,p)}{\partial n}ds$$
$$=\int_{\partial V_{m'}(p)} G_{V_m}(z,q)\frac{\partial G(z,p)}{\partial n}ds. \qquad (2)$$

Put $G(z,q)-G_{V_m}(z,q)=H(z)$. Then $H(z)$ is positive and vanishes almost everywhere on $\partial V_m(p)$ $\left(\text{with respect to the measure of } \frac{\partial G(z,p)}{\partial n}ds\right)$.

Let $H_{V_{m'}}(z)$ be the lower envelope non negative continuous superharmonic functions in $V_m(p)$ which are larger than $H(z)$ in $V_m(p)-V_{m'}(p)$: $m'>m$. Then

$$G_{V_{m'}}(z,q)=G_{V_m}(z,q)+H_{V_{m'}}(z). \quad \text{Hence by (2)}$$
$$G_{V_m}(p,q)\geqq G_{V_m}(p,q),$$

where
$$G_{V_m}(p,q)=\frac{1}{2\pi}\int_{\partial V_m(p)} G(z,q)\frac{\partial G(z,p)}{\partial n}ds.$$

We define the value of $G(z,q)$ at p, denoted by $G(p,q)$, by $\lim_{m=\infty} G_{V_m}(p,q)$.

When $q\in R$, this $G(p,q)$ is the same that is defined before.

We shall prove the following

Theorem 1. 1) *$G(p,p)=\infty$.*

2) *$G(z,p)$ is $\mathfrak{M}$-lower semicontinuous in $\overline{R}$.*

3) *$G(z,p)$ is superharmonic in weak sense.*[7]

4) *$G(p,q)=G(q,p)$.*

1) is clear by Lemma 2 and 3) is also clear by definition of $G(q,p)$.

Proof of 2). Let $p_j\overset{\mathfrak{M}}{\to}p$. Put $G^M_{V_m}(p,q)=\frac{1}{2\pi}\int_{\partial V_m(p)}\tilde{G}^M(z,q)\frac{\partial G(z,p)}{\partial n}ds$, then there exists n, for every positive number ε, such that

$$G^M_{V_m}(p,q)\leqq\frac{1}{2\pi}\int_{\partial V_m(p)\cap R_n}\tilde{G}^M(z,q)\frac{\partial G(z,p)}{\partial n}ds+\varepsilon.$$

Since the genus of $R_{n+1}-R_0$ is finite, map $R_{n+1}-R_0$ onto a compact surface on the w-plane. $(R_n-R_0)\cap\partial V_m(p)$ is composed of at most a finite number of analytic curves. We make sufficiently narrow strip B in $R_{n+1}-R_0$ such that B contains $\partial V_m(p)\cap R_n$, and ∂B passes end points of $\partial V_m(p)\cap R_n$ orthogonaly. We divide B into a finite number

7) If $U(p)\geqq\frac{1}{2\pi}\int_C U(z)\frac{\partial G(z,p)}{\partial n}ds$ for only the niveau curve C of the Green's function with pole at p, we say that $U(z)$ is superharmonic in weak sense.

of narrow strips B_l $(l=1, 2, \cdots, k)$ so that ∂B_l intersect $\partial V_m(p)$ with angles $(\neq 0, \pi)$ and we map B_l onto a rectangle: $0 \leqq Im\,\zeta \leqq \delta$ (δ is sufficiently small), $-1 \leqq Re\,\zeta \leqq 1$, on the ζ-plane such that any vertical straight line: $Re\,\zeta = s$: $-1 \leqq s \leqq 1$ intersects only once $\partial V_m(p_j)$: $j > j_0$. This is possible, since $G(z, p_j) \to G(z, p)$, and their derivatives converge. We make a point α_j of $\partial V_m(p_j)$ correspond to a point α of $\partial V_m(p)$, where $Re\,\alpha_j = Re\,\alpha$. Since $\frac{\partial G(\alpha, p_j)}{\partial n}\,ds \geqq 0$ and uniformly bounded in B_l and since $\frac{G(\alpha_j, p_j)}{\partial n}\,ds \to \frac{\partial G(\alpha, p)}{\partial n}\,ds$ and since $G(\alpha_j, q) \to G(\alpha, q)$, we have

$$\lim_{j=\infty} \int_{B_l \cap \partial V_m(p_j)} \tilde{G}^M(\alpha_j, q) \frac{\partial G(\alpha_j, p_j)}{\partial n}\,ds = \int_{B_l \cap \partial V_m(p)} \tilde{G}^M(\alpha, q) \frac{\partial G(\alpha, p)}{\partial n}\,ds,$$

whence $\displaystyle \lim_{j=\infty} \int_{B \cap \partial V_m(p_j)} \tilde{G}^M(z, q) \frac{\partial G(z, p_j)}{\partial n}\,ds = \int_{B \cap \partial V_m(p)} \tilde{G}^M(z, q) \frac{\partial G(z, p)}{\partial n}\,ds$, and

$$\lim_{j=\infty} G^M_{V_m(p_j)}(p_j, q) = \lim_{j=\infty} \int_{\partial V_m(p_j)} \tilde{G}^M(z, q) \frac{\partial G(z, p_j)}{\partial n}\,ds \geqq \lim_{j=\infty} \int_{\partial V_m(p_j) \cap B} \tilde{G}^M(z, q) \frac{\partial G(z, p_j)}{\partial n}\,ds$$

$$= \int_{\partial V_m(p)} \tilde{G}^M(z, q) \frac{\partial G(z, p)}{\partial n}\,ds - \varepsilon. \quad \text{Let } \varepsilon \to 0. \quad \text{Then}$$

$$\varliminf_{j} G^M_{V_m(p_j)}(p_j, q) \geqq G^M_{V_m(p)}(p_j, q). \quad \text{Hence}$$

$G^M_{V_m}(p, q)$ is $\mathfrak{M}$-lower semicontinuous.

If $p_j \in B$, we consider $p_{j_i} \in R$ such that $\lim_i p_{j_i} = p_j$.

Since $G^M_{V_m}(p, q) \uparrow G_{V_m}(p, q)$ and since $G_{V_m}(p, q) \uparrow G(p, q)$, $G(z, q)$ is also $\mathfrak{M}$-lower semicontinuous at p, whence $G(z, q)$ is lower semicontinuous in $\bar{R}$ (not only in R where $G(z, q)$ is continuous).

Proof of 4). Let ξ and η be points of R lying on $\partial V_m(p)$ and $\partial V_n(q)$ respectively. If η is outside of $V_m(p)$,

$$G(p, \eta) = G(\eta, p) = \frac{1}{2\pi} \int_C G(z, \eta) \frac{\partial G(z, p)}{\partial n}\,ds.$$

If $\eta \in V_m(p)$,

$$\frac{1}{2\pi} \int_C G(z, \eta) \frac{\partial G(z, p)}{\partial n}\,ds = m \leqq G(\eta, p) = G(p, \eta),$$

where $C = \partial V_m(p)$.

Since $G_{V_m(p)}(p, q) = \frac{1}{2\pi} \int_C G(\xi, q) \frac{\partial G(\xi, p)}{\partial n}\,ds$, and since $V_n(q) \to B$, when $n \to \infty$, for any given positive number ε, there exists a niveau curve $C' = V_n(q)$ such that

$$G_{V_m(p)}(p, q) - \varepsilon \leqq \frac{1}{2\pi} \int_{\underline{C}} G(\xi, q) \frac{\partial G(\xi, p)}{\partial n}\,ds,$$

where $\underline{C}$ is the part of C out of $V_n(q)$. Let ξ be a point on $\underline{C} \cap R$.

Then $p \notin V_n(q)$, whence

$G(\xi, q)=G(q, \xi)=\dfrac{1}{2\pi}\displaystyle\int_{C'} G(\eta, \xi)\frac{\partial G(\eta, q)}{\partial n}ds.$ Accordingly we have

$$G_{V_{m}(p)}(p, q)-\varepsilon \leqq \frac{1}{4\pi^2}\int_{\underline{C}}\left(\int_{C'} G(\eta, \xi)\frac{\partial G(\eta, q)}{\partial n}ds\right)\frac{\partial G(\xi, p)}{\partial n}ds$$

$$=\frac{1}{4\pi^2}\int_{C'}\left(\int_{\underline{C}} G(\xi, \eta)\frac{\partial G(\xi, p)}{\partial n}ds\right)\frac{\partial G(\eta, q)}{\partial n}ds.$$

If $\eta \notin V_m(p)$

$$\frac{1}{2\pi}\int_{\underline{C}} G(\xi, \eta)\frac{\partial G(\xi, p)}{\partial n}ds \leqq \frac{1}{2\pi}\int_{C} G(\xi, \eta)\frac{\partial G(\xi, p)}{\partial n}ds \leqq G(\eta, p)=G(p, \eta).$$

If $\eta \in V_m(p)$

$$\frac{1}{2\pi}\int_{\underline{C}} G(\xi, \eta)\frac{\partial G(\xi, p)}{\partial n}ds \leqq \frac{1}{2\pi}\int_{C} G(\xi, \eta)\frac{\partial G(\xi, p)}{\partial n}ds \leqq G(p, \eta)=G(\eta, p).$$

On the other hand $G_{V_{n}(q)}(q, p)=\dfrac{1}{2\pi}\displaystyle\int_{C'} G(\eta, p)\frac{\partial G(\eta, q)}{\partial n}ds.$ Hence

$$G_{V_{m}(p)}(p, q)-\varepsilon \leqq \frac{1}{4\pi^2}\int_{C'}\left(\int_{C} G(\xi, \eta)\frac{\partial G(\xi, p)}{\partial n}ds\right)\frac{\partial G(\eta, q)}{\partial n}ds$$

$$=\frac{1}{2\pi}\int_{C'} G(\eta, p)\frac{\partial G(\eta, q)}{\partial n}ds = G_{V_{n}(q)}(q, p).$$

Since the inverse inequality holds for the other $V_{m'}(p)$ and $V_{n'}(q)$ and since $G_{V_m(p)}(p, q) \uparrow G(p, q)$ and $G_{V_m(q)}(q, p) \uparrow G(q, p)$, we have 4).