26. *Capacity of Subsets of the Ideal Boundary*

By Zenjiro KURAMOCHI
Mathematical Institute, Osaka University
(Comm. by K. KUNUGI, M.J.A., Feb. 13, 1956)

In the previous paper,[1] we introduced the notion of the capacity of the subset of the ideal boundary and proved some theorems. Unfortunately their proofs were much complicated. The purpose of the present article is to give simple proofs. Let R be a Riemann surface with a positive boundary. Let $\{R_n\}$ $(n=0,1,2,\cdots)$ be an exhaustion of R with compact relative boundaries $\{\partial R_n\}$[2] and D be a non compact domain in R whose relative boundary ∂D is composed of at most enumerably infinite number of analytic curves clustering nowhere in R. We say that a sequence $\{D\cap(R-R_n)\}$ determines a subset B_D of the ideal boundary.

1. Capacity of a Subset B_D. Let $U_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-R_0-(D\cap(R_{n+i}-R_n))$ (in short we denote it by $B_{n,n+i}$) such that $U_{n,n+i}(z)=0$ on ∂R_0, $U_{n,n+i}(z)=1$ on $(\partial R_n\cap D)+(\partial D\cap(R_{n+i}-R_n))$ and $\dfrac{\partial U_{n,n+i}(z)}{\partial n}=0$ on $\partial R_{n+i}-D$. Then we have the Dirichlet's integral

$$\underset{B_{n,n+i}}{D}(U_{n,n+i}(z)-U_{n,n+i+j}(z),\ U_{n,n+i}(z))=0,$$

whence

$$\underset{B_{n,n+i}}{D}(U_{n,n+i+j}(z))=\underset{B_{n,n+i}}{D}(U_{n,n+i}(z))+\underset{B_{n,n+i}}{D}(U_{n,n+i}(z)-U_{n,n+i+j}(z)). \qquad (1)$$

But it is easily seen by Dirichlet's principle that $\underset{B_{n,n+i}}{D}(U_{n,n+i}(z))\leqq \underset{R_1-R_0}{D}(U^*(z))\leqq M<\infty$ for every n and i, where $U^*(z)$ is a harmonic function in R_1-R_0 such that $U^*(z)=0$ on ∂R_0 and $U^*(z)=1$ on ∂R_1. Therefore by (1)

$$M\geqq \underset{B_{n,n+i+j}}{D}(U_{n,n+i+j}(z))\geqq \underset{B_{n,n+i}}{D}(U_{n,n+i+j}(z))\geqq \underset{B_{n,n+i}}{D}(U_{n,n+i}(z)),$$

hence the sequence $\{\underset{B_{n,n+i+j}}{D}(U_{n,n+i+j}(z))\}$ is convergent, which implies $\lim\limits_{\substack{i=\infty\\ j=\infty}}\underset{B_{n,n+i+j}}{D}(U_{n,n+i+j}(z)-U_{n,n+i}(z))=\lim\limits_{\substack{i=\infty\\ j=\infty}}\{\underset{B_{n,n+i+j}}{D}(U_{n,n+i+j}(z))-\underset{B_{n,n+i}}{D}(U_{n,n+i}(z))\}=0.$ Thus $\{U_{n,n+i}(z)\}$ converges to a function $U_n(z)$ in mean. Since $U_{n,n+i}(z)=0$ on ∂R_0, it converges uniformly in every compact set of $R-(D\cap(R-R_n))$. We see $U_{n+i,n+i+j}(z)\leqq U_{n,n+i+j}(z)$, by the maximum principle. From this we have $\lim\limits_{j=\infty}U_{n+i,n+i+j}(z)=U_{n+i}(z)\leqq U_n(z)=\lim\limits_{j=\infty}$

1) Z. Kuramochi: Harmonic measures and capacity of sets of the ideal boundary. I–II, Proc. Japan Acad., **30-31** (1954–1955).

2) In this article, we denote by ∂A the relative boundary of A.

$U_{n,n+i+j}(z)$, hence $\{U_n(z)\}$ converges to a function $U(z)$. We shall prove that $\{U_n(z)\}$ converges to $U(z)$ in mean. Since

$$\underset{B_{n,n+i+j}}{D}(U_{n,n+i+j}(z), U_{n+i,n+i+j}(z)) = \int_{\partial(D\cap(R_{n+i+j}-R_n))} U_{n,n+i+j}(z)\frac{\partial U_{n+i,n+i+j}(z)}{\partial n}ds$$

$= \underset{B_{n,n+i}}{D}(U_{n+i,n+i+j}(z))$, $\underset{B_{n,n+i+j}}{D}(U_{n,n+i+j}(z)-U_{n+i,n+i+j}(z)) = \underset{B_{n,n+i+j}}{D}(U_{n,n+i+j}(z)) - \underset{B_{n,n+i+j}}{D}(U_{n+i,n+i+j}(z))$. Let $j\to\infty$. Then we see that $\{\underset{R-((R-R_n)\cap D)}{D}(U_n(z))\}$ is decreasing. Hence $D(U_n(z))$ is convergent, whence $D(U_n(z) - U_{n+i}(z))\to 0$, if n and $i\to\infty$. Therefore $\{U_n(z)\}$ converges to $U(z)$ in mean. Put $\lim_{n=\infty} D(U_n(z)) = \int_{\partial R_0}\frac{\partial U(z)}{\partial n}ds = \mathrm{Cap}(B_D)$. We call it the *capacity* of B_D and $U(z)$ the *equilibrium potential* of B_D.

In what follows, we show that $U(z)$ has the essential properties of the equilibrium potential in space.

Lemma 1. Let G be a domain containing a non compact domain D. Let $\{U_\alpha(z)\}$ be the family of harmonic functions with the boundary value φ on $\partial R_0+\partial D$. In this family, there exists a harmonic function with the boundary value φ and has the minimal Dirichlet's integral.[3] *Let this function be $U_D(z)$. Let $U_G(z)$ be a harmonic function in $R-R_0-G$ with the boundary value $U_D(z)$ on $\partial G+\partial R_0$ such that $U_G(z)$ has the minimal Dirichlet's integral over $R-R_0-G$. Then*

$$U_D(z)\equiv U_G(z).$$

Let $U_n'(z)$ be a harmonic function in R_n-R_0-G such that $U_n'(z) = U_D(z)$ on $\partial G+\partial R_0$ and $\frac{\partial U_n'}{\partial n}=0$ on $\partial R_n - G$. Then we see, as before, that $\{U_n'(z)\}$ converges to $U'(z)$ in mean and $U'(z)$ has the minimal Dirichlet's integral among all functions with boundary value $U_D(z)$ on $\partial R_0+\partial G$. If $\underset{R-R_0-G}{D}(U'(z)) \leqq \underset{R-R_0-G}{D}(U_D(z)) - d$ $(d>0)$, then $\underset{R_n-R_0-G}{D}(U_n'(z)) \leqq \underset{R-R_0-G}{D}(U_D(z)) - d$ $(n=1,2,\cdots)$. Now let $U_n''(z)$ be a harmonic function in R_n-R_0-D such that $U_n''(z)=U_D(z)$ on $\partial R_n\cap(G-D)+\partial R_0$ and $U_n''(z)=U'(z)$ on $\partial R_n - G$. Then by Dirichlet's principle $\underset{R_n-R_0-D}{D}(U_n''(z)) \leqq \underset{R-R_0-G}{D}(U_n'(z)) + \underset{(R_n-R_0)\cap(G-D)}{D}(U_D(z)) \leqq \underset{R_n-R_0-D}{D}(U_D(z)) - d$. Choose a subsequence $\{U_{n'}''(z)\}$ of $\{U_n''(z)\}$ which converges uniformly in $R-R_0-D$ to $U^*(z)$. Then we have also $\underset{R-R_0-D}{D}(U^*(z)) \leqq \varliminf_{n=\infty} \underset{R_n-R_0-D}{D}(U_n''(z)) \leqq \underset{R-R_0-D}{D}(U_D(z)) - d$. This contradicts the minimality of $\underset{R-R_0-D}{D}(U_D(z))$. Hence $\underset{R-R_0-G}{D}(U_D(z)) = \underset{R-R_0-G}{D}(U'(z))$. The function $U'(z)$ is clearly the harmonic continuation of $U_D(z)$ by Dirichlet's principle. On the

3) In the present article, we suppose that there is at least one function with bounded Dirichlet's integral in this family.

other hand, it is clear that such $U'(z)$ is determined uniquely by the boundary value on $\partial R_0+\partial G$.[4] Hence $U'(z)\equiv U_D(z)$.

Theorem 1. Let $\hat{U}_n(z)$ $(n=1,2,\cdots)$ be the harmonic function in $R-R_0(D\cap(R-R_n))$ such that $\hat{U}_n(z)=U(z)$ on $\partial(D\cap(R-R_n))$, $\hat{U}_n(z)=0$ on ∂R_0 and $\hat{U}_n(z)$ has the minimal Dirichlet's integral. Then

$$\hat{U}_n(z)\equiv U(z).$$

Proof. Since $(D\cap(R-R_n))\supset(D\cap(R-R_{n+i}))$, by Lemma 1, $U_{n+i}(z)$ $(n=1,2,\cdots)$ has the minimal Dirichlet's integral over $R-R_0-(D\cap(R-R_n))$ among all functions with the boundary value $U_{n+i}(z)$ on $\partial R_0+\partial(D\cap(R-R_n))$, hence

$$D(U_{n+i}(z)\pm\varepsilon V_n(z))$$
$$=D(U_{n+i}(z))\pm 2\varepsilon D(U_{n+i}(z),\ V_n(z))+\varepsilon^2 D(V_n(z))\geqq D(U_{n+i}(z))$$

and

$$D(U_{n+i}(z),\ V_n(z))=0$$

for any small positive number ε, where $V_n(z)$ is a harmonic function in $R-R_0-(D\cap(R-R_n))$ such that $V_n(z)=0$ on $\partial R_0+\partial(D\cap(R-R_n))$ and $D(V_n(z))<\infty$.

Since $\{U_{n+i}(z)\}$ converges to $U(z)$ in mean,

$$0=\lim_{i=\infty} D(U_{n+i}(z)-U(z),\ V_n(z))\leqq\lim_{i=\infty}\sqrt{D(U_{n+i}(z)-U(z))D(V_n(z))}=0.$$

Hence $$D(U(z),\ V_n(z))=0.$$

Since $V_n(z)$ is arbitrary, $U(z)$ has the minimal Dirichlet's integral over $R-R_0-(D\cap(R-R_n))$, whence $\hat{U}_n(z)=U(z)$.

Corollary 1. If $U(z)\not\equiv 0$, $\overline{\lim\limits_{z\in D}}\, U(z)=1$.

Let $\hat{U}_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-R_0-(D\cap(R_{n+i}-R_n))$ such that $\hat{U}_{n\,n+i}(z)=0$ on ∂R_0, $\hat{U}_{n,n+i}(z)=U(z)$ on $\partial(D\cap(R_{n+i}-R_n))$ and $\dfrac{\partial\hat{U}_{n,n+i}(z)}{\partial n}=0$ on $\partial R_{n+i}-D$. Then by Theorem 1, $\lim\limits_{i=\infty}\hat{U}_{n,n+i}(z)=U(z)$ for every n. Suppose $U(z)\leqq K<1$ in D. Then by maximum principle $\hat{U}_{n,n+i}(z)\leqq KU_{n,n+i}(z)$. Let $i\to\infty$ and $n\to\infty$. Then $U(z)\leqq KU(z)$. This is absurd. This completes the proof.

Denote by J_λ $(\lambda<1)$ the domain in which $U(z)\leqq\lambda$. Put $H_\lambda=D\cap J_\lambda$. Then H_λ is a non compact domain[5] which determines a subset B_λ.

Corollary 2. B_λ is a set of capacity zero.

Let $U_\lambda(z)$ be the equilibrium potential of B_λ. Then it is clear that $U_\lambda(z)\leqq U(z)$. Hence $\overline{\lim\limits_{z\in H_\lambda}}\, U_\lambda(z)\leqq\lambda<1$. This contradicts Corollary 1, therefore $U_\lambda(z)\equiv 0$.

4) If there were two harmonic functions above-mentioned, by the minimality of $D(U_i(z))$ $(i=1,2)$, we have $D(U_i(z)),\ \pm\varepsilon(U_1(z)-U_2(z))\geqq D(U_i(z))$ for every small positive number ε and $D(U_i(z),\ U_1(z)-U_2(z))=0$, whence $D(U_1(z)-U_2(z))=0$. Thus $U_1(z)\equiv U_2(z)$.

5) In what follows, we suppose that H_λ and D_λ are non compact. If they are compact, our assertion is trivial.

Lemma 2. Let D_1 and D_2 be two non compact domains. Then D_1+D_2 is also a non compact domain. Then

$$\mathrm{Cap}\,(B_{D_1})+\mathrm{Cap}\,(B_{D_2})\geqq \mathrm{Cap}\,(B_{D_1+D_2}),$$

where B_{D_1}, B_{D_2} and $B_{D_1+D_2}$ are subsets of ideal boundary determined by D_1, D_2 and D_1+D_2 respectively.

From our definition this is evident.

2. On the Behaviour of the Green's Function in the Neighbourhood of the Ideal Boundary. Let $\{R_n\}$ $(n=1,2,\cdots)$ be an exhaustion of R. Let $G(z,p)$ be the Green's function of R with pole at p and let M be so large number that $G_M=\underset{z\in R}{\mathcal{E}}[G(z,p)\geqq M]$ is compact. We can suppose $G_M=R_0$. If we consider $R-R_0$ as a non compact domain D which defines the ideal boundary of R. Then it is clear that $1-\dfrac{G(z,p)}{M}=U(z)$. Put $D_\lambda=\underset{z\in R}{\mathcal{E}}[G(z,p)>\lambda]$ $(\lambda>0)$. Then D_λ is a domain determining a subset B_λ of the boundary which we call irregular set. Then by Lemma 2, we have the following

Theorem 2. The irregular set of the Green's function is of capacity zero.

Let $w_n^p(z)$ be a harmonic function in $D_\lambda\cap R_n$ such that $w_n^p(z)=0$ on ∂D_λ and $w_n^p(z)=1$ on $\partial R_n\cap D_\lambda$. Then $w_n^p(z)\downarrow w^p(z)$. M. Parreau[6] proved that $w^p(z)=0$. Let $w_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-((R_{n+i}-R_n)\cap D)$ such that $w_{n\,n+i}(z)=0$ on $\partial R_{n+i}-D$ and $w_{n,n+i}(z)=1$ on $\partial(D\cap(R_{n+i}-R_n))$. Put $\lim\limits_{n=\infty}\lim\limits_{i=\infty} w_{n,n+i}(z)=w(z)$. Let $w'_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-R_0-((R_{n+i}-R_n)\cap D)$ such that $w'_{n,n+i}(z)=0$ on $\partial R_0+\partial R_{n+i}-D$ and $w'_{n,n+i}(z)=1$ on $\partial(D\cap(R_{n+i}-R_n))$. Put $\lim\limits_{n=\infty}\lim\limits_{i=\infty} w'_{n\,n+i}(z)=w'(z)$. We proved that $w'(z)=0$ is equivalent to $w(z)=0$.[7] Then we have $0=U(z)\geqq w'(z)$ which implies $0=w(z)\geqq w^p(z)$. Hence the theorem is an extension of F. Vasilesco and contains the result of Parreau.[8]

We can construct an open Riemann surface $\hat{D}_\lambda$ by the process of symmetrization with respect to ∂D_λ. Then we have the following

Corollary. $D_\lambda+\hat{D}_\lambda$ is a null-boundary Riemann surface.

Proof. Let $\omega_n(z)$ be the harmonic measure of $(\partial R_n\cap D_\lambda)+(\widehat{\partial R_n\cap D_\lambda})$ with respect to $((D_\lambda\cap R_n)-R_0)+(\widehat{(D_\lambda\cap R_n)-R_0})$. Then $\omega_n(z)=0$ on $\partial R_0+\partial\hat{R}_0$, $\omega_n(z)=1$ on $(\partial R_n\cap D_\lambda)$ and $\dfrac{\partial\omega_n}{\partial n}=0$ on ∂D_λ. On the other hand, let $U_{n,n+i}(z)$ be a function in $R_{n+i}-((R_{n+i}-R_n)\cap D_\lambda)-R_0$ such

6) M. Parreau: Sur les moyennes des fonctions harmoniques et la classification des surfaces de Riemann, Annales de l'Institute Fourier (1952).

7) See 1).

8) The set of irregular points of the Green's function in space is of capacity zero.

that $U_{n,n+i}(z)=0$ on ∂R_0, $U_{n,n+i}(z)=1$ on $\partial(D_\lambda\cap(R_{n+i}-R_n))$ and $\frac{\partial U_{n,n+i}(z)}{\partial n}=0$ on $\partial R_{n+i}-D_\lambda$. Then it is clear that $\underset{(D_\lambda\cap R_n)-R_0}{D}(\omega_n(z))\leqq\underset{R_{n+i}-R_0}{D}(U_{n,n+i}(z))$. Hence, since B_D is a set of capacity zero, we have $\underset{D_\lambda\cap(R-R_0)}{D}(\lim_{n=\infty}\omega_n(z))\leqq\underset{R-R_0}{D}(\lim_{n=\infty}\lim_{i=\infty}U_{n,n+i}(z))=0$. It follows that $\lim_{n=\infty}\omega_n(z)=0$. Thus $D_\lambda+\hat{D}_\lambda$ is a Riemann surface with a null-boundary.

3. Capacity with respect to a non Compact Domain. Let D_1 and D_2 be two non compact domains in R such that $D_1\subset D_2$. Let $U_{n,n+i}(z)$ be a harmonic function in $(R_{n+i}\cap D_2)-(D_1\cap(R_{n+i}-R_n))$ such that $U_{n,n+i}(z)=0$ on $\partial D_2\cap R_{n+i}$, $U_{n,n+i}(z)=1$ on $(\partial R_n\cap D_1)+D_1\cap(R_{n+i}-R_n)$ and $\frac{\partial U_{n,n+i}}{\partial n}=0$ on $\partial R_{n+i}\cap(D_2-D_1)$. If $D(U_{n,n+i}(z))\leqq M<\infty$ for a certain n and every i, we can prove, as before, that $\{U_{n,n+i}(z)\}$ tends to $U_n(z)$ in mean and $U_n(z)\to U(z)$ in mean. We call $D(U(z))$ the capacity of B_{D_1} determined by D_1 with respect to D_2. We used these results to prove that O_{HD} is invariant by a quasi-conformal mapping whose dilatation quotient is bounded.[9]

4. Correction to the Previous Paper. We used the following lemma in the previous paper.[10] *Let $U(z)$ be the harmonic function in $R-R_0-(D\cap(R-R_n))$ such that $U(z)=0$ on ∂R_0, $U(z)=1$ on $(D\cap(R-R_n))$ and $U(z)$ has the minimal Dirichlet's integral. Put $G_\varepsilon=\underset{z\in R}{\varepsilon}[U(z)>1-\varepsilon]$. Then*

$$\int_{\partial G_\varepsilon}\frac{\partial U(z)}{\partial n}ds=\int_{\partial R_0}\frac{\partial U(z)}{\partial n}ds$$

for every ε except at most one ε'.

But the proof was not complete. We prove, instead of the above, the following lemma. *There exists a set H in the open interval* $(0,1)$ *such that $\mathrm{mes}\,H=1$ and if* $\varepsilon\in H$, then

$$L=\int_{\partial R_0}\frac{\partial U(z)}{\partial n}ds=\int_{\partial G_\varepsilon}\frac{\partial U(z)}{\partial n}ds.$$

Proof. Let $U'_m(z)$ be a harmonic function in $R_m-R_0-G_\varepsilon$ such that $U'_m(z)=0$ on ∂R_0, $U'_m(z)=1-\varepsilon$ on ∂G_ε and $\frac{\partial U'_m}{\partial n}=0$ on $\partial R_m-G_\varepsilon$. Then by Lemma 1, $\lim_{m=\infty}U'_m(z)=U(z)$. On the other hand, since $\int_{R_m\cap\partial G_\varepsilon}\frac{\partial U'_m}{\partial n}ds=\int_{\partial R_0}\frac{\partial U'_m}{\partial n}ds$ and $\frac{\partial U'_m}{\partial n}\geqq 0$ on ∂G_ε, $\lim_{m=\infty}\int_{\partial G_\varepsilon}\frac{\partial U'_m}{\partial n}ds=\int_{\partial R_0}\lim_{m=\infty}\frac{\partial U'_m}{\partial n}ds$. We

9) Z. Kuramochi: On the existence of harmonic functions on Riemann surfaces, Osaka Math. Journ., **7** (1955).

10) ——: Harmonic measures and capacity of sets of the ideal boundary. I, Proc. Japan Acad., **30** (1954).

have by Fatou's lemma $L_\varepsilon = \int\limits_{\partial G_\varepsilon} \frac{\partial U}{\partial n} ds \leqq \lim\limits_{m=\infty} \int\limits_{\partial R_0} \frac{\partial U'_m}{\partial n} = L$. We can take $p+iq \equiv U(z)+iV(z)$ as the local parameter at every point of $R-R_0-(D \cap (R-R_n))$, where $V(z)$ is the conjugate function $U(z)$. Then

$$\frac{\partial U(z)}{\partial p}=1, \quad \frac{\partial U(z)}{\partial q}=0 \text{ on } \partial G_\varepsilon,$$

and

$$L=D(U(z))=\int\int\left\{\left(\frac{\partial U(z)}{\partial p}\right)^2+\left(\frac{\partial U(z)}{\partial q}\right)^2\right\}dpdq=\int_0^1 q_\varepsilon dp,$$

where $q_\varepsilon = \int\limits_{\partial G_\varepsilon} \frac{\partial U}{\partial n} ds$. Suppose that there was a set E of positive measure such that if $\varepsilon \in E$, q_ε is smaller than L. We have $D(U_n(z)) < L$. This is absurd. This completes the proof. In the previous paper[11] we used the fact that there exists a dense set F in $(0, 1)$ such that if $\varepsilon \in F$, then $\int\limits_{\partial G_\varepsilon} \frac{\partial U}{\partial n} ds = L$. Thus the proof of the theorem in the previous paper[11] is valid.

11) See 10).