41. A Note on Banach Algebras

By Yasue MIYANAGA

(Comm. by K. KUNUGI, M.J.A., March 12, 1956)

R. E. Edwards [1] has shown the following

Theorem. If R is a complex Banach algebra with unit element and if the norm satisfies the condition $\|x^{-1}\|=\|x\|^{-1}$ *for all* $x\in R$, *then R is isomorphic to the complex field.*

In this Note, we shall give a proof of the theorem mentioned above. Our proof is essentially due to E. Hille [2].

Proof. Let $\lambda e-x$ be a regular element in R. Since, for λ in the resolvent set $\rho(x)$, the inverse of $\lambda e-x$ exists, the condition of the norm shows that

$$\|(\lambda e-x)^{-1}\|=\|\lambda e-x\|^{-1} \qquad \text{in } \rho(x).$$

From the formula

$$(\lambda e-x)^{-1}=(\lambda_0 e-x)^{-1}\left\{e+\sum_{n=1}^{\infty}(\lambda_0-\lambda)^n[(\lambda_0 e-x)^{-1}]^n\right\}.$$

We see that if $(\lambda_0 e-x)^{-1}$ exists, then the series is absolutely convergent when $|\lambda-\lambda_0|<\|\lambda_0 e-x\|$, therefore $(\lambda e-x)^{-1}$ exists for

$$|\lambda-\lambda_0|<\|\lambda_0 e-x\|.$$

Hence, we may continue $(\lambda e-x)^{-1}$ analytically along any path starting at $\lambda=\lambda_0$ on which $\lambda e-x\neq\theta$.

There can be at most one value of λ for which $\lambda e-x$ vanishes; on the other hand, there must be at least one such value since $(\lambda e-x)^{-1}$, which is holomorphic at infinity, can not be holomorphic for all finite values of λ.

Hence there exists a complex number ζ such that $x=\zeta e$. This completes the proof of the theorem.

References

[1] R. E. Edwards: Multiplicative norms on Banach algebra, Proc. Cambridge Philos. Soc., **47**, 473-474 (1951).

[2] E. Hille: Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., New York, **31** (1945).