39. *Notes on Topological Spaces. II. Some Properties of Topological Spaces with Lebesgue Property*

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In this Note, we suppose that all spaces considered are separated. It is well known that every continuous function on a compact space is uniformly continuous. An elegant and interesting generalisation of it may be found in W. Sierpinski's classic work [4]. We shall prove the following

Theorem 1. Let $f(x)$ and $g(x)$ be, respectively, upper and lower semi-continuous (real valued) functions on a uniform space S with countable Lebesgue property,) and suppose that $f(x)\leq g(x)$ for all x in S. Then, for any positive ε, there is a surrounding V such that*

$$f(x')<g(x'')+\varepsilon$$

for $x', x'' \in V(x)$ of every x in S.

Proof. From $f(x)\leq g(x)$, we have

$$f(x)<g(x)+\varepsilon$$

for all x in S. By the assumption of $f(x)$, $g(x)$,

$$O_r=\{x \mid f(x)<r,\ r<g(x)+\varepsilon\}$$

is open set for every rational r. The family $F=\{O_r \mid r: \text{rational}\}$ is clearly an open covering of S. On the other hand, since S has countable Lebesgue property, we can find a surrounding V such that each $V(x)$ is contained in some $O_r \in F$. Hence, if $x', x'' \in V(x)$, x', x'' are contained in some $O_r \in F$. Therefore $x', x'' \in V(x)$ implies $f(x')<r$ and $r<g(x)+\varepsilon$, and hence if $x', x'' \in V(x)$, we have

$$f(x')<g(x'')+\varepsilon.$$

The proof is complete.

If we take $f(x)=g(x)$ in Theorem 1, we have the following

Corollary 1. If any covering of a uniform space S has countable Lebesgue property, every continuous function on S is uniformly continuous.

Since compact space has Lebesgue property, we have two corollaries.

Corollary 2. Let $f(x)$ and $g(x)$ be, respectively, upper and lower semi-continuous functions on a compact space S, and suppose that $f(x)\leq g(x)$ for all x in S. Then, for any positive ε, there is a surrounding V such that

$$f(x')<g(x'')+\varepsilon$$

*) For terminologies, see K. Iséki [1].

for $x', x'' \in V(x)$.

Corollary 3. Let $f(x)$ *and* $g(x)$ *be, respectively, upper and lower semi-continuous functions on a compact metric space* S, *and suppose that* $f(x) \leq g(x)$ *for all* x *in* S. *Then, to every number* $\varepsilon > 0$, *corresponds a number* δ *such that* $\rho(x', x'') < \delta$ *implies* $f(x') < g(x'') + \varepsilon$.

Corollary 3 is essentially due to W. Sierpiński [4].

Concerning a metric space with the Lebesgue property, we shall give a simple proof of a theorem. A theorem we shall prove is a special case of a theorem by S. Kasahara [2].

A. A. Monteiro and M. M. Peixoto [3] proved the following

Proposition. The following properties of a metric space S *are equivalent:*

(1) *Finite open covering of* S *has Lebesgue property.*

(2) *Countable open covering of* S *has Lebesgue property.*

(3) *Any open covering of* S *has Lebesgue property.*

(4) *Every continuous function on* S *is uniformly continuous.*

(5) *Every bounded continuous function on* S *is uniformly continuous.*

We shall prove that, if a metric space S is connected, then each of these properties implies the compactness of S.

Theorem 2. If every continuous function on a connected metric space S *is uniformly continuous, then* S *is compact.*

Proof. To prove Theorem 2 it is sufficient to show that S is sequentially compact.

Suppose that S is not sequentially compact. Then there are countable points x_n $(n=1, 2, \cdots)$ having no limit point.

Therefore there are spherical neighbourhoods $S_n = \{x \mid \rho(x_n, x) < \varepsilon_n\}$ such that

1) $$\bar{S}_m \frown \bar{S}_n = 0 \quad (m \neq n,\ m, n = 1, 2, \cdots)$$

2) $$\varepsilon_n \downarrow 0.$$

We shall define a continuous function $f(x)$ on S as follows:

$$f(x) = \begin{cases} n\left\{1 - \dfrac{\rho(x, x_n)}{\varepsilon_n}\right\}, & \text{for } x \in \bar{S}_n, \\ 0, & \text{for } x \in S - \bigcup_{n=1}^{\infty} \bar{S}_n. \end{cases}$$

The function $f(x)$ is clearly continuous.

Since S is connected, every sphere $S' = S(x_n, \varepsilon)$ $(0 < \varepsilon < \varepsilon_n)$ meets S. Hence $f(x)$ on S_n contains all numbers between 0 and n. Therefore there are two points x', x'' such that $\rho(x', x'') < \dfrac{1}{n}$ implies $|f(x') - f(x'')| > n$. This shows that $f(x)$ is not uniformly continuous, which contradicts the hypothesis. Hence S is compact. The proof is complete.

References

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[4] W. Sierpiński: Sur une propriété des fonctions semi-continues, Fund. Math., **9**, 1–2 (1927).