36. *On the Efficiency of Leontief's Dynamic Input-Output System*

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In this paper we shall show that the usual Leontief dynamic input-output system can be characterized as an intertemporally efficient path[1] in a more general economic model.

1. We shall consider a closed economy with n industries. Let $x_i(t)$ be the total output of industry i, $x_{ik}(t)$ be the flow inputs of the product of industry i used in the production of $x_k(t)$ in the period $[t, t+1]$; and let $s_i(t)$ be the total stock of the i-th capital, $s_{ik}(t)$ be the stock of the i-th capital being used by industry k at the point of time t. Then the *generalized Leontief dynamic input-output system* for the period $[t, t+1]$ can be written as follows:

$$(1)\qquad \begin{aligned} & x_i(t)=\sum_{k=1}^{n} x_{ik}(t)+(s_i(t+1)-s_i(t)),\\ & s_i(t)=\sum_{k=1}^{n} s_{ik}(t),\ \ s_i(t+1)=\sum_{k+1}^{n} s_{ik}(t+1),\\ & x_{ik}(t)\geqq a_{ik}x_k(t),\ \ s_{ik}(t+1)\geqq b_{ik}x_k(t),\\ & s_i(t)\geqq 0,\ \ s_i(t+1)\geqq 0,\\ & \qquad (i, k=1,\cdots, n), \end{aligned}$$

where a_{ik} are input coefficients and b_{ik} are stock coefficients:

$$(2)\qquad \begin{aligned} & a_{ik}\geqq 0,\ \ b_{ik}\geqq 0,\\ & \sum_{k=1}^{n} a_{ik}\leqq 1, \quad (i, k=1,\cdots, n),\\ & \sum_{k=1}^{n} a_{ik}<1 \quad \text{for at least one } i. \end{aligned}$$

Let

$$(3)\qquad \begin{aligned} & x(t)=\begin{bmatrix} x_1(t)\\ \vdots \\ x_n(t)\end{bmatrix},\ \ X(t)=[x_{ik}(t)],\\ & s(t)=\begin{bmatrix} s_1(t)\\ \vdots \\ s_n(t)\end{bmatrix},\ \ S(t)=[s_{ik}(t)],\\ & A=[a_{ik}],\ \ B=[b_{ik}]. \end{aligned}$$

Then (1) may be rewritten as follows:[2]

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1) For "intertemporal efficiency", cf. Samuelson [4] or Uzawa [6].

2) We denote, for two vectors x, y,

$$\begin{aligned} & x\geqq y, \text{ when } x_i\geqq y_i\ (i=1,\cdots, n),\\ & x\geq y, \text{ when } x\geqq y \text{ and } x\neq y,\\ & x>y, \text{ when } x_i>y_i\ (i=1,\cdots, n). \end{aligned}$$

$$(4)\quad \begin{aligned} & x(t)=X(t)\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}+s(t+1)-s(t),\\ & s(t)=S(t)\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix},\quad s(t+1)=S(t+1)\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix},\\ & X(t)\geqq A\begin{bmatrix}x_1(t) & & 0\\ & \ddots & \\ 0 & & x_n(t)\end{bmatrix},\quad S(t+1)\geqq B\begin{bmatrix}x_1(t) & & 0\\ & \ddots & \\ 0 & & x_n(t)\end{bmatrix},\\ & s(t)\geqq 0,\quad s(t+1)\geqq 0.\end{aligned}$$

2. A pair (s^0, s^1) of non-negative vectors s^0, s^1 is said to be a *possible pair*, if there exist $x(t), X(t), S(t)$, and $S(t+1)$ such that they satisfy (4) with $s(t)=s^0$ and $s(t+1)=s^1$. If (s^0, s^1) is a possible pair and there is no non-negative vector s for which (s^0, s) is possible and $s\geq s^1$, then (s^0, s^1) is said to be an *efficient pair*.

For any possible pair (s^0, s^1), there is a possible pair $(s^0, \bar{s}^1)$ so that $\bar{s}^1\geqq s^1$ and (4) are satisfied with

$$(5)\quad \begin{aligned} & s^0=s(t),\quad \bar{s}^1=s(t+1),\\ & s(t+1)=s(t)+(I-A)x(t)\\ & \bar{X}(t)=A\begin{bmatrix}x_1(t) & & 0\\ & \ddots & \\ 0 & & x_n(t)\end{bmatrix},\quad \ddot{S}(t+1)\geqq B\begin{bmatrix}x_1(t) & & 0\\ & \ddots & \\ 0 & & x_n(t)\end{bmatrix}.\end{aligned}$$

Let

$$T(s^0)=\{s^1;\ s^1=s^0+(I-A)x\geqq 0,\ (B+A-I)x\leqq s^0\}$$

and $E(s^0)$ be the set of efficient vectors in $T(s^0)$.[3] Then (s^0, s^1) is possible for any $s^1\in T(s^0)$, and $E(s^0)$ coincides with the set of a vector s^1 for which (s^0, s^1) is an efficient pair $T(s^0)$ is a closed convex polyhedral set for any non-negative vector s^0.

3. A sequence of vectors $(s^0, s^1, \cdots, s^h)$ is said to be a *possible sequence*, if $s^{t+1}\in T(s^t)$ for any $t=0,\cdots,h-1$. For any non-negative vector s^0, we shall denote by $T^h(s^0)$ the set of a vector s^h, for which there are $h-1$ vectors $s^1,\cdots,s^{h-1}$ such that $(s^0, s^1,\cdots,s^{h-1}, s^h)$ is a possible sequence. $T^h(s^0)$ is a closed convex polyhedral set for any non-negative vector s^0 and any integer h.

A possible sequence $(s^0,\cdots,s^h)$ is said to be *h-efficient*, if s^h is efficient in $T^h(s^0)$. A vector s^h is efficient in $T^h(s^0)$, if and only if $s^h\in T^h(s^0)$ and there is a positive vector $p>0$ such that

$$p\cdot s^h=\max_{s\in T^h(s^0)} p\cdot s.^{4)}$$

4. Let $T^*(s^1)=\{s^0;\ s^1\in T(s^0)\}$. Then $T^*(s^1)$ coincides with the

3) A vector $\bar{x}$ in a set T is said to be efficient if $\bar{x}\leq x$ for no $x\in T$.

4) Cf. Arrow-Barankin-Blackwell [1]. $p\cdot s$ stands for the inner product of vectors p and s.

set of all vectors s^0 for which there is a vector x^0 such that

$$s^0 = s^1 - (I-A)x^0 \geqq 0, \quad Bx^0 \leqq s^1.$$

Therefore, if a sequence $(s^0, \cdots, s^h)$ is h-efficient, the subsequence $(s^t, \cdots, s^{t+k})$ is k-efficient for any $0 \leqq t \leqq h-1, 1 \leqq k \leqq h-t$. On the other hand, a possible sequence (s^0, s^1, s^2) is 2-efficient if and only if there is a positive vector $p>0$ such that

(a) $p \cdot s^1$ maximizes $p \cdot s$ subject to $s \geqq 0$ and $(C-I)s \leqq Cs^0$,
(b) $p \cdot s^1$ minimizes $p \cdot s$ subject to $s \geqq 0$ and $Cs \geqq (C-I)s^2$;
where $C = B(I-A)^{-1}$.[5)]

5. We shall consider the case where B and $(C-I)$ are non-singular and $C(C-I)^{-1} \geqq 0$. In this case, the usual Leontief dynamic system can be characterized by the intertemporal efficiency.

Theorem 1. *Let B and $(C-I)$ be non-singular, and $C(C-I)^{-1} \geqq 0$. Then a possible sequence (s^0, s^1, s^2) is 2-efficient, if and only if there are vectors x^0 and x^1 such that*

$$\text{(6)} \qquad \begin{aligned} s^1 &= s^0 + (I-A)x^0, \quad (B+A-I)x^0 = s^0 \\ s^1 &= s^2 - (I-A)x^1, \quad Bx^1 = s^2. \end{aligned}$$

From this theorem we can deduce the following

Theorem 2.[6)] *Let B and $(C-I)$ be non-singular, $C(C-I)^{-1} \geqq 0$ and $s^0 \in [B+A-I] = \{(B+A-I)x; x \geqq 0\}$. Then $(s^0, s^1, \cdots, s^h)$ is h-efficient if and only if (s^t, s^{t+1}, s^{t+2}) is 2-efficient for any $t = 0, 1, \cdots, h-2$.*

6. Suppose the Frobenius root[7)] ρ of C be greater than 1. Then ρ is also the Frobenius root of $(I-A)^{-1}B$, and there exists a non-zero vector x^0 such that

$$(I-A)^{-1}Bx^0 = \rho x^0, \quad x^0 \geqq 0.$$

If we set $s^0 = (B+A-I)x^0$ and $s^1 = Bx^0$, then

$$s^1 = \lambda s^0, \quad s^0 \geqq 0, \quad s^1 \geqq 0,$$

where $\lambda = \dfrac{\rho}{\rho-1} > 1$. Hence we get the existence of the balanced rate of growth.[8)]

Theorem 3. *If the Frobenius root ρ of $C = B(I-A)^{-1}$ is greater than 1, then there exists a possible sequence $(\bar{s}^0, \bar{s}^1, \cdots, \bar{s}^h)$ such that*

$$\text{(7)} \qquad \bar{s}^{t+1} = \lambda \bar{s}^t \quad (t = 0, 1, \cdots, h-1)$$

for any integer h, where $\lambda = \dfrac{\rho}{\rho-1} > 1$.

5) Cf. Uzawa [6].

6) Cf. Samuelson [4] and Uzawa [6].

7) The Frobenius root ρ of a matrix C with non-negative elements is the largest non-negative characteristic root of C, which has a maximum absolute value among characteristic roots of C. $[\lambda I - C]^{-1} \geqq 0$, if and only if $\lambda > \rho$. Cf. Debreu-Herstein [2].

8) Cf. Solow-Samuelson [5].

In the case where $C(C-I)^{-1} \geqq 0$, the Frobenius root of C is greater than 1 and the sequence $(\bar{s}^0, \bar{s}^1, \cdots, \bar{s}^h)$ in the above theorem is h-efficient for any integer h. It also becomes the equilibrium point in the sense of von Neumann [3].

References

[1] K. J. Arrow, E. W. Barankin, and D. Blackwell: Admissible points of convex sets, Contributions to the Theory of Games, **2**, 87–91 (1953).
[2] G. Debreu and I. N. Herstein: Non-negative square matrices, Econometrica, **21**, 597–607 (1953).
[3] J. von Neumann: Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes, Ergebnisse eines math. Kolloquiums, No. 8, 73–83 (1935–1936).
[4] P. A. Samuelson: Efficient paths of dynamic capital development (to be published).
[5] R. M. Solow and P. A. Samuelson: Balanced growth and constant returns to scale, Econometrica, **21**, 412–424 (1953).
[6] H. Uzawa: On the efficiency conditions of capital accumulation, Ann. Inst. Statist. Math., **7** (1956).