35. On Galois Theory of Division Rings

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Recently N. Nobusawa[1] has succeeded to generalize Krull's Galois theory for fields of infinite degree to division rings. But in his paper he discussed separately the compact Galois group and the discrete Galois group. In our investigation, considering the locally compact case, we can discuss the above two cases at the same time, moreover the discussion will present a generalization of Nobusawa's theory. In this note, we shall state our results without proofs.[2]

1. Preliminaries. Let K be a division ring and L be a division subring of K. If $\mathfrak{H}$ is a group of automorphisms of K, the fixed subring of $\mathfrak{H}$ in K will be denoted as $J(\mathfrak{H}, K)$, and, for any division subring D, $\mathfrak{H}(D)$ means the totality of D-automorphisms in $\mathfrak{H}$.

In case K is Galois over L, the maximal group $\mathfrak{G}=\mathfrak{G}(K/L)$ is said to be *locally finite-dimensional* (abbreviated, l.f.d.) if, for each finite subset S of K, $L(S^{\mathfrak{G}})$ (the least division subring containing L and $S^{\mathfrak{G}}$) is finite over L (as a left L-module). When the maximal group $\mathfrak{G}$ is l.f.d., we can introduce a Hausdorff topology in it by the same manner as in Nobusawa's theory,[3] and then it becomes a topological group.

For notations, $V_K(M)$ signifies the centralizer of M in K, $\widetilde{M}$ the totality of inner automorphisms determined by non-zero elements of M, and $\mathfrak{M}_S$ means the restriction of $\mathfrak{M}$ to S, where S is a subset of K and $\mathfrak{M}$ a set of mappings of K.

We state here two lemmas, the former of which is a generalization of Cartan's theorem[4] and the latter is due to N. Jacobson.[5]

Lemma 1. *Let R, S be division subrings of a division ring D. If each inner automorphism determined by an element of S leaves R*

1) N. Nobusawa: An extension of Krull's Galois theory to division rings, Osaka Math. J., **7** (1955).

2) The details will appear under the same article in Math. J. Okayama Univ., **6**, No. 1.

3) The maximal group $\mathfrak{G}=\mathfrak{G}(K/L)$ is said to be *locally finite* when, for any a in K, $\{a\}^{\mathfrak{G}}$ is finite. Nobusawa's theory is constructed under the assumption that $\mathfrak{G}$ is locally finite, and there a fundamental neighbourhood system of the identity of $\mathfrak{G}$ is defined as the totality of $\mathfrak{G}(N)$, where N runs over all division subrings which are finite and normal over L (i.e. $N^{\sigma}=N$ for all σ in $\mathfrak{G}$).

4) R. Brauer: On a theorem of H. Cartan, Bull. Amer. Math. Soc., **55** (1949).

5) This result will appear in the forthcoming book of Jacobson.

set-wise invariant, then $R \supset S$ *or* $R \subset V_D(S)$.

Lemma 2. *Let* $\mathfrak{H}$ *be a regular group of* K/L,[6] H *be an intermediate division subring with* $[H:L]<\infty$. *If* σ *is an* L*-isomorphism of* H *into* K, *then it can be extended to an element in* $\mathfrak{H}$.

2. Local finite-dimensionality. Throughout this section we assume that K is Galois and locally finite over L (i.e. $L(S)$ is finite over L for any finite subset S of K), and $\mathfrak{G}$ means the maximal group $\mathfrak{G}(K/L)$.

Our first easy result is the next

Theorem 1. (i) *If* $\mathfrak{G}$ *is l.f.d., then the following conditions are equivalent to each other:*

(1) $\mathfrak{G}$ *is compact.*

(2) $\mathfrak{G}$ *is locally finite.*

(3) $\mathfrak{G}$ *is almost outer, that is,* $\mathfrak{G}$ *contains only a finite number of inner automorphisms.*

(ii) *In case* $\mathfrak{G}$ *is l.f.d.,* $\mathfrak{G}$ *is discrete if and only if* K *is finite over* L.

We consider here the following conditions:

(I) For each $a \in K$, the subspace spanned by $\{a\}^{\mathfrak{G}}$ over L is finite over L.

(I′) For each $a \in K$, the subspace spanned by $\{a\}^{\mathfrak{J}}$ over L is finite over L, where $\mathfrak{J}$ is the totality of inner automorphisms in $\mathfrak{G}$.

(II) $[V_K(L):V_L(L)]<\infty$.

(II′) $[L(V_K(L)):L]<\infty$.

(III) $V_K(L)=V_K(K)$.

Clearly (III) is nothing but to say that $\mathfrak{G}$ is outer and (I) is equivalent to that $\mathfrak{G}$ is l.f.d.

By making use of Lemma 1, we can prove the next

Lemma 3. *If* $V_K(L) \supsetneqq V_K(K)$, *then (I) implies (II).*

From the above lemma, we obtain the following

Theorem 2. *(I)* $\Leftrightarrow$ *(I′)* $\Leftrightarrow$ *(II) or (III) ((II′) or (III)).*

In the rest of this section, we shall restrict our attention to the case where $\mathfrak{G}$ is l.f.d., and set $H=V_K(V_K(L))$.

Lemma 4. *Let* $\mathfrak{G}$ *be l.f.d., and non-outer,* S *be a finite subset of* K. *Then there exists a subring* T *with the following properties:*

(1) T *contains* $H(S)$ *and normal over* L.

(2) $[T:H]<\infty$.

(3) $[V_T(L):V_T(T)]<\infty$.

In fact, let N be a subring containing $L(S, V_K(L))$ which is finite and normal over L. Then $T=V_K(V_N(N))$ is a desired one.

6) $\mathfrak{H}$ is called a regular group of K/L, if it contains all the inner automorphisms determined by elements of $V_K(L)$ and $J(\mathfrak{H}, K)=L$.

From the preceding lemma we obtain the following

Theorem 3. *If* $\mathfrak{G}$ *is l.f.d., then so is* $\mathfrak{G}(K/H)$ *and* $\widetilde{V_K(L)}=\mathfrak{G}(K/H)$, *where* $\widetilde{V_K(L)}$ *is the topological closure of* $\widetilde{V_K(L)}$ *in* $\mathfrak{G}$.

By making use of Theorem 3, the following is given.

Theorem 4. *In case* $\mathfrak{G}$ *is l.f.d.,* $\mathfrak{G}$ *is locally compact if and only if* $[V_K(L):V_K(K)]<\infty$.

We shall conclude this section by giving the following theorem concerning some special case.

Theorem 5. *Let* $\mathfrak{G}$ *be l.f.d., and non-outer. If* $[L:V_L(L)]<\infty$, *then* $[H:L]<\infty$.

Combining this fact with the latter part of Theorem 3, we may say roughly that, in this case, $\mathfrak{G}$ is outer or almost inner.

3. A generalization of Nobusawa's theory. Throughout this section, we assume again that K is Galois over L and denote by $\mathfrak{G}$ the maximal group $\mathfrak{G}(K/L)$. At first, we shall deal with the more (strictly) general case, and come back later to the locally compact case. Our assumptions, which should be satisfied if $\mathfrak{G}$ is l.f.d. and locally compact, are the following:

(α) $H=V_K(V_K(L))$ is locally finite over L.
(β) $[V_K(L):V_K(K)]<\infty$.
(γ) $L_1=L(d_1,\cdots,d_n)$ is finite over L, where $\{d_1,\cdots,d_n\}$ is a (fixed) H-basis of K.
(δ) K is Galois over L_1.

Our principal results are the following two theorems. The former enables us to prove the existence of the fundamental correspondence between intermediate subrings and subgroups which are maximal in Nobusawa's sense, and the latter is the so-called extension theorem, of which the proof is given by Lemma 2 and that of the former.

Theorem 6. *Under the assumptions* (α)—(δ), *for any intermediate subring* K', *there holds that* $J(\mathfrak{G}(K'),K)=K'$ *and* K *is locally finite over* K'.

Theorem 7. *Under the assumptions* (α)—(δ), *each* L*-isomorphism of an arbitrary intermediate subring* K' *into* K *can be extended to an automorphism in* $\mathfrak{G}$.

To prove Theorem 6, the following lemma plays an essential rôle.

Lemma 5. *Under the assumptions* (α)—(δ), *there hold the following:*

(i) *K is locally finite over L.*
(ii) *There exists a one-to-one correspondence between subrings* H_2 *of*

H with $[H_2:H_1]<\infty$ *and subrings* L_2 *of* K *with* $[L_2:L_1]<\infty$ *in the relations* $H_2=H\cap L_2$ *and* $L_2=\sum_{i=1}^{n}\oplus H_2d_i$, *where* $H_1=H\cap L_1$.

(iii) $\mathfrak{G}(H/L)=\mathfrak{G}_H$.

To be noted, in (ii) of Lemma 5, if H_2 is normal over H_1, then L_2 is normal over L_1, and conversely. This fact shows that the mapping $\varphi:\sigma\to\sigma_H$ (the restriction of σ on H) of the compact group $\mathfrak{G}(K/L_1)$ into $\mathfrak{G}(H/H_1)$ is continuous, which together with the fact that $\mathfrak{G}(K/L_1)_H$ is dense in $\mathfrak{G}(H/H_1)$ implies $\mathfrak{G}(K/L_1)_H=\mathfrak{G}(H/H_1)$. The proof of (iii) is given in virtue of this fact and Lemma 2.

The next is only an easy consequence of Lemma 5, but the proof of Theorem 6 can be reduced to it.

Corollary. *Under the assumptions* $(\alpha)-(\delta)$, *if* $H\supset L'\supset L$, *then* $J(\mathfrak{G}(L'),K)=L'$.

To the above end, the following two lemmas are required.

Lemma 6. *Under the assumptions* $(\alpha)-(\delta)$, *there exists a subgroup* $\mathfrak{G}_0$ *of* $\mathfrak{G}$ *such that* $J(\mathfrak{G}_0,K)=L$ *and such that, for each finite subset* S *of* K, $[L(S^{\mathfrak{G}_0}):L]<\infty$.

Lemma 7. *Let* $\mathfrak{G}_0$ *be a group of automorphisms of a division ring* D, *and* $Z=J(\mathfrak{G}_0,D)$. *If, for each finite subset* S *of* D, $[Z(S^{\mathfrak{G}_0}):Z]<\infty$, *then* $J(\mathfrak{G}(D'),D)=D'$ *for any division subring* D' *such that* $[D':Z]<\infty$, *where* $\mathfrak{G}=\mathfrak{G}(D/Z)$.

To prove Theorem 6, it suffices, for any intermediate subring K', to choose a subring K'' finite over L such that $K''\subset K'\subset V_K(V_K(K''))$, what is possible by (β).

Now, we shall come back to the case where $\mathfrak{G}$ is l.f.d. and locally compact. From the latter part of Theorem 6 we obtain the next

Lemma 8. *If* $\mathfrak{G}$ *is l.f.d. and locally compact, then any regular closed subgroup of* $\mathfrak{G}$ *is a maximal group in Nobusawa's sense, and conversely.*

And we obtain the following

Theorem 8. *If* $\mathfrak{G}$ *is l.f.d. and locally compact, then there exists a one-to-one correspondence between regular closed subgroups of* $\mathfrak{G}$ *and intermediate subrings, in the usual sense of Galois theory.*

By Theorem 7 and Lemma 8, the next theorem is given.

Theorem 9. *We set the same assumptions as in Theorem 8 and, for an intermediate subring* K', *we set* $\mathfrak{K}=\{\sigma\in\mathfrak{G};K'^{\sigma}=K'\}$. *Then* K' *is Galois over* L *if and only if the composite of* $\mathfrak{K}$ *and* $\widehat{V_K(J(\mathfrak{K},K))}$ *is dense in* $\mathfrak{G}$.