56. On Semi-reducible Measures. II

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In this note we show that main results concerning semi-reducibility of Baire (Borel) measures, which have been proved by Marczewski and Sikorski [5] in metric spaces, and by Katětov [4, Theorem 1] and the present author [3, Theorem 4] in paracompact spaces, are valid in completely regular spaces with a complete structure.[1] The case of two-valued measures has already been considered by Shirota [6], though his result is related to Q-spaces of Hewitt [1]. We use the same notations as in the previous paper [3]: $\mathfrak{B}^*(X)$=all of Baire subsets in a T-space $X, C(X, R)$=all of real-valued continuous functions on $X, P(f)=\{x \mid f(x)>0, f \in C(X, R)\}$, $\mathfrak{P}(X)=\{P(f) \mid f \in C(X, R)\}$.

All spaces considered are completely regular spaces and all measures considered are finite measures, unless the contrary is explicitly stated.

Lemma 1. *If any closed discrete subset in a T_1-space X has the power of (two-valued) measure 0,[2] then for any (two-valued) Baire measure μ in X, the union of a discrete collection of open subsets $\{G_\alpha \mid G_\alpha \in \mathfrak{P}(X), \mu(G_\alpha)=0\}$ has also μ-measure 0.*[3]

Since the proof is essentially stated in the previous paper [3, Theorem 4], we do not repeat it here.

Lemma 2. *Let $\mathfrak{U}=\{U_\alpha \mid \alpha \in A\}$ be a normal covering of a T-space X. Then there exists a refinement $\mathfrak{V}=\{G_{n\alpha} \mid \alpha \in A, n=1,2,\cdots\}$ of $\mathfrak{U}$ such that $\{G_{n\alpha} \mid \alpha \in A\}$ is a discrete collection with $G_{n\alpha} \in \mathfrak{P}(X)$ for each n.*

Proof. Let $\mathfrak{U}=\{U_\alpha \mid \alpha \in A\}$ be a normal covering of X and let $\{\mathfrak{U}_n\}$ be a normal sequence such that $\mathfrak{U}_1 \overset{\Delta}{>} \mathfrak{U}_2 \overset{\Delta}{>} \cdots \overset{\Delta}{>} \mathfrak{U}_n \overset{\Delta}{>} \cdots$. Then, as Stone [7] has showed, there exists a closed covering $\{F_{n\alpha} \mid \alpha \in A, n=1,2,\cdots\}$ satisfying the following conditions:

i) $S(F_{n\alpha}, \mathfrak{U}_{n+3}) \cap S(F_{n\gamma}, \mathfrak{U}_{n+3})=\phi$ if $\alpha \neq \gamma$,

ii) $\{F_{n\alpha} \mid \alpha \in A\}$ is a discrete collection for each n,

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1) A measure μ defined on a σ-field $\mathfrak{B}$ containing Baire family in a T-space is called semi-reducible if there exists a closed subset Q such that (1) $\mu(G)>0$ holds if G is open, $G \in \mathfrak{B}$, $G \frown Q \neq \phi$, and (2) $\mu(F)=0$ holds if F is closed, $F \in \mathfrak{B}$, $F \frown Q=\phi$.

2) A discrete set is called to have the power of (two-valued) measure 0, if every (two-valued) measure, defined for all subsets and vanishing for all one point, vanishes identically.

3) A collection $\{H_\alpha \mid \alpha \in A\}$ of subsets of a T-space is called discrete if (1) the closures $\overline{H}_\alpha$ are mutually disjoint, (2) $\cup_{\beta \in B} \overline{H}_\beta$ is closed for anv subset B of A.

iii) $S(F_{n\alpha}, \mathfrak{U}_{n+1}) \subset U_\alpha$

Let $\{f_{n\alpha}\}$ be continuous functions as follows: $f_{n\alpha}(x)=1$ if $x \in F_{n\alpha}$, $f_{n\alpha}(x)=0$ if $x \notin S(F_{n\alpha}, \mathfrak{U}_{n+4})$, $0 \leqq f_{n\alpha}(x) \leqq 1$ otherwise. Set $G_{n\alpha}=\{x \mid f_{n\alpha}(x)>0\}$. Then, since $G_{n\alpha} \subset S(F_{n\alpha}, \mathfrak{U}_{n+4})$, we have $\overline{G}_{n\alpha} \subset S(G_{n\alpha}, \mathfrak{U}_{n+4}) \subset S(S(F_{n\alpha}, \mathfrak{U}_{n+4}), \mathfrak{U}_{n+4}) \subset S(F_{n\alpha}, \mathfrak{U}_{n+3})$. We shall show that $F=\bigcup_{\alpha \in B} \overline{G}_{n\alpha}$ is closed in X for any subset B of A. Suppose that $p \in \overline{F}$. Then every neighborhood $N(p)$ of p meets some $\overline{G}_{n\alpha}(\alpha \in B)$ and so meets some $G_{n\alpha}(\alpha \in B)$. If $N(p)$ is contained in $S(p, \mathfrak{U}_{n+4})$, we have $p \in S(G_{n\alpha}, \mathfrak{U}_{n+4}) \subset S(F_{n\alpha}, \mathfrak{U}_{n+3})$. This shows that every neighborhood $N(p)$ of p contained in $S(G_{n\alpha}, \mathfrak{U}_{n+4})$ meets only one $G_{n\alpha}$. Thus it holds that $p \in \overline{G}_{n\alpha}$, i.e., $p \in F$. Hence $\mathfrak{B}=\{G_{n\alpha} \mid \alpha \in A,\ n=1,2,\cdots\}$ is a refinement of $\mathfrak{U}$ satisfying the necessary conditions.

Lemma 3. *In a space X the following conditions are equivalent:*

(1) *any (two-valued) Baire measure in X which is locally measure 0 has total measure 0,*[4]

(2) *any (two-valued) Baire measure in X is semi-reducible.*

Proof. (1)$\rightarrow$(2). Let μ be a (two-valued) Baire measure and let $Q(\mu)=\{p \mid \mu(U_p)>0$ for any neighborhood $U_p \in \mathfrak{B}^*(X)$ of $p\}$.[5] If $Q(\mu)=\phi$, we have $\mu(X)=0$ by (1) and so μ is obviously semi-reducible. Therefore, we can suppose that $Q(\mu) \neq \phi$. In the case when μ is a two-valued measure the subset $Q(\mu)$ contains only one point, and hence it is trivial that μ is semi-reducible. In general case we show that $\mu(F)=0$ is valid for any closed subset F such that $F \in \mathfrak{B}^*(X)$ and $F \frown Q(\mu)=\phi$ hold. For this purpose we define a Baire measure ν as follows:

$$\nu(B)=\mu(B \frown F) \quad \text{for any Baire subset } B \text{ of } X.$$

Then, since ν is locally measure 0, we obtain $\nu(X)=\mu(F)=0$. (2)$\rightarrow$(1). Let μ be a (two-valued) Baire measure in X which is locally measure 0. Since μ is semi-reducible by the hypothesis, there exists a closed subset Q such that 1) $\mu(G)>0$ holds if G is open, $G \in \mathfrak{B}^*(X)$, $G \frown Q \neq \phi$, and 2) $\mu(F)=0$ holds if F is closed, $F \in \mathfrak{B}^*(X)$, $F \frown Q=\phi$. But the closed subset Q must be a null set, for μ is locally measure 0. Hence we obtain $\mu(X)=0$ by 2).

Lemma 4. *If any (two-valued) Baire measure in a space X which is locally measure 0 has total measure 0, then any closed discrete subset in X has the power of (two-valued) measure 0.*

Proof. Let ν be a (two-valued) Borel measure in a closed discrete subset $Y=\{p_\alpha\} \subset X$ vanishing at each point p_α. We define a (two-valued) Baire measure μ in X as follows:

4) A Baire measure μ is called locally measure 0, if for any point $p \in X$ there is a neighborhood $U_p \in \mathfrak{B}^*(X)$ of p with $\mu(U_p)=0$.

5) The subset $Q(\mu)$ is obviously closed in X.

$\mu(B)=\nu(B\frown Y)$ for any Baire subset B of X.

Since μ is obviously locally measure 0, we obtain $\mu(X)=\nu(Y)=0$, which shows that Y has the power of (two-valued) measure 0.

Theorem 1. *Let X be a space with a complete structure. Then the following conditions are equivalent:*

(1) *any closed discrete subset of X has the power of measure 0,*

(2) *for any Baire measure μ in X, the union of discrete collection of open subsets $\{G_\alpha \mid G_\alpha \in \mathfrak{P}(X), \mu(G_\alpha)=0\}$ has also μ-measure 0,*

(3) *any Baire measure in X which is locally measure 0 has total measure 0,*

(4) *any Baire measure in X is semi-reducible.*

Proof. (1)$\rightarrow$(2), (3)$\rightleftarrows$(4) and (3)$\rightarrow$(1) follow from Lemmas 1, 3 and 4 respectively. Hence we shall prove only (2)$\rightarrow$(3). Let μ be a Baire measure in X which is locally measure 0, and suppose that $\mu(X)>0$ holds. On the other hand, let gX be a complete structure of X and let $\{\mathfrak{U}_\lambda \mid \lambda \in \Lambda\}$ be the uniformity of gX. Then, by Lemma 2, there exists an open refinement $\mathfrak{B}_\lambda=\{H^\lambda_{n\alpha} \mid \alpha \in A_\lambda,\ n=1,2,\cdots\}$ of $\mathfrak{U}_\lambda$ such that $\{H^\lambda_{n\alpha} \mid \alpha \in A_\lambda\}$ is a discrete collection with $H^\lambda_{n\alpha} \in \mathfrak{P}(X)$ for each n, since $\mathfrak{U}_\lambda$ is a normal covering. Hence by (2) we have $\mu(H^\lambda_{n\alpha})>0$ for some n and α. Moreover by the transfinite induction, we can show that it is possible to choose an $H_\lambda \in \mathfrak{B}_\lambda$ for any $\lambda \in \Lambda$ such that $\mu(\bigcap_{i=1}^\infty H_{\lambda_i})>0$ holds for any $\lambda_i \in \Lambda$ $(i=1,2,\cdots)$. Let Λ be well ordered and let normal coverings $\{\mathfrak{U}_\lambda \mid \lambda \in \Lambda\}$ be denoted by $\mathfrak{U}_1, \mathfrak{U}_2, \cdots, \mathfrak{U}_\lambda, \cdots$. For a normal covering $\mathfrak{U}_1$, we can take an $H_1 \in \mathfrak{B}_1$ such that $\mu(H_1)>0$ holds. Now fix a $\lambda_0 \in \Lambda$ and suppose that for any $\nu<\lambda_0$ we can choose $H_\nu \in \mathfrak{B}_\nu$ such that $\mu(\bigcap_{i=1}^\infty H_{\nu_i})>0$ for any $\nu_i<\lambda_0$ $(i=1,2,\cdots)$. Since there exists at most countable number of $H^\lambda_{n\alpha} \in \mathfrak{B}_\lambda$ with $\mu(H^\lambda_{n\alpha})>0$ for each λ, we denote them as $\{H^\lambda_i\}$ $(i=1,2,\cdots)$. Now put $E_\lambda=\bigcup_{n,\alpha}\{H^\lambda_{n\alpha} \mid \mu(H^\lambda_{n\alpha})=0\}$. $\{H^\lambda_{n\alpha}\}$ being a discrete open collection for any λ and n, it follows from (2) that $\mu(E_\lambda)=0$. Since $X=E_\lambda \smile (\bigcup_{i=1}^\infty H^\lambda_i)$, we have $\mu(\bigcup_{i=1}^\infty H^\lambda_i)=\mu(X)$. If it were impossible to choose an $H_{\lambda_0} \in \mathfrak{U}_{\lambda_0}$ such that $\mu((\bigcap_{i=1}^\infty H_{\nu_i})\frown H_{\lambda_0})>0$ for any $\nu_i<\lambda_0$ $(i=1,2,\cdots)$, there would exist $\{\nu_{ij}\}$ $(j=1,2,\cdots)$ for any $H^{\lambda_0}_i$ satisfying the following conditions:

i) $$\nu_{ij}<\lambda_0$$

ii) $$\mu((\textstyle\bigcap_{j=1}^\infty H_{\nu_{ij}})\frown H^{\lambda_0}_i)=0.$$

Then it would hold that

$$\mu((\textstyle\bigcap_{i,j} H_{\nu_{ij}})\frown(\bigcup_i H^{\lambda_0}_i))=0.$$

This contradicts the following facts: $\mu(\bigcap_{i,j} H_{\nu_{ij}})>0$ and $\mu(\bigcup_i H^{\lambda_0}_i)=\mu(X)$. Hence the induction is completed. Then $\{H_\lambda \mid \lambda \in \Lambda\}$ is obviously a Cauchy family of gX. Therefore, there exists a point $p \in X$ such that any neighborhood $U_p \in \mathfrak{B}^*(X)$ of p contains some H_λ. Since $\mu(H_\lambda)>0$ for any $\lambda \in \Lambda$, any neighborhood $U_p \in \mathfrak{B}^*$ of p has positive μ-measure. This contradicts the fact that μ is locally measure 0. Thus we have $\mu(X)=0$, which completes the proof.

Concerning two-valued measures, we have the following results.

Theorem 2. *Let X be a space with a complete structure. Then the following conditions are equivalent:*

(1) *any closed discrete subset in X has the power of two-valued measure 0,*

(2) *for any two-valued Baire measure μ in X, the union of discrete collection of open subsets $\{G_\alpha \mid G_\alpha \in \mathfrak{P}(X), \mu(G_\alpha)=0\}$ has also μ-measure 0,*

(3) *any two-valued Baire measure in X which is locally measure 0 has total measure 0,*

(4) *any two-valued Baire measure in X is semi-reducible.*

Proof. It is sufficient to prove only (2)$\rightarrow$(3). Let μ be a two-valued Baire measure in X which is locally measure 0, and suppose that $\mu(X)=1$ holds. Let gX be a complete structure of X, let $\{\mathfrak{U}_\lambda \mid \lambda \in \Lambda\}$ be the uniformity of gX and let $\mathfrak{V}_\lambda=\{H^\lambda_{n\alpha} \mid \alpha \in A_\lambda, n=1, 2,\cdots\}$ be an open refinement of $\mathfrak{U}_\lambda$ such that $\{H^\lambda_{n\alpha} \mid \alpha \in A_\lambda\}$ is a discrete collection with $H^\lambda_{n\alpha} \in \mathfrak{P}(X)$ for each n. Then we can choose an $H_\lambda \in \mathfrak{V}_\lambda$ with $\mu(H_\lambda)=1$ for any λ. Since it is obvious that $\{H_\lambda \mid \lambda \in \Lambda\}$ has the finite intersection property, $\{H_\lambda \mid \lambda \in \Lambda\}$ is a Cauchy family. Therefore there is a point $p \in X$ such that any neighborhood $U_p \in \mathfrak{B}^*(X)$ of p contains some H_λ. Thus any neighborhood $U_p \in \mathfrak{B}^*(X)$ of p has positive μ-measure. This contradicts the fact that μ is locally measure 0. Hence we have $\mu(X)=0$, which completes the proof.

Since a space which has the property (4) in Theorem 2 is a Q-space [2, Theorem 16], we have the following corollary, which has been shown by Shirota [6].

Corollary. *Let X be a space with a complete structure. Then the following conditions are equivalent:*

(1) *X is a Q-space,*

(2) *any closed discrete subset in X is a Q-space.*

Remark. We note that we can replace Baire measures with Borel measures and finite measures with σ-finite measures in theorems stated above.

References

[1] E. Hewitt: Rings of real-valued continuous functions, Trans. Amer. Math. Soc., **64**, 45–99 (1948).

[2] E. Hewitt: Linear functionals on spaces of continuous functions, Fund. Math., **37**, 161–189 (1950).

[3] T. Ishii: On semi-reducible measures, Proc. Japan Acad., **31**, 648–652 (1955).

[4] M. Katětov: Measures in fully normal spaces, Fund. Math., **38**, 73–84 (1951).

[5] E. Marczewski and R. Sikorski: Measures in non separable metric spaces, Coll. Math., **1**, 133–139 (1948).

[6] T. Shirota: A class of topological spaces, Osaka Math. J., **4**, 23–40 (1952).

[7] A. H. Stone: Paracompactness and product spaces, Bull. Amer. Math. Soc., **54**, 977–982 (1948).