55. On a Relation between Dimension and Metrization

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(Comm. by K. KUNUGI, M.J.A., April 12, 1956)

The purpose of the present note is to show that the dimension of a metrizable space is defined by some characters of metrics which agree with the topology of the space.

In this note we concern ourselves only with metrizable spaces and take the definition of dimension by H. Lebesgue or the equivalent definition: $\dim \phi=-1$, $\dim R \leqq n$ if for any pair of a closed subset F and an open subset G with $F \subseteq G$ there exists an open set U such that $F \subseteq U \subseteq G$, $\dim(\bar{U}-U) \leqq n-1$.[1]

We state here a theorem previously proved by the author[2] which will be needed in the proof of our main theorem.

In order that a T_1 topological space R is a metrizable space with $\dim R \leqq n$ it is necessary and sufficient that there exists a sequence $\mathfrak{B}_1 > \mathfrak{B}_2^* > \mathfrak{B}_2 > \mathfrak{B}_3^* > \cdots$ of open coverings such that $S(p, \mathfrak{B}_m)$ $(m=1, 2, \cdots)$[3] is a nbd (=neighbourhood) basis for each point p of R and such that each set of $\mathfrak{B}_{m+1}$ intersects at most $n+1$ sets of $\mathfrak{B}_m$.

Theorem. *In order that $\dim R \leqq n$ for a metrizable space R it is necessary and sufficient that one can assign a metric $\rho(x, y)$ agreeing with the topology of R such that for every $\varepsilon>0$ and for every point x of R, $\rho(S_{\varepsilon/2}(x), y_i)<\varepsilon$ $(i=1, \cdots, n+2)$ imply $\rho(y_i, y_j)<\varepsilon$ for some i, j with $i \neq j$.*[4]

Proof. Necessity. 1. Let R be a metrizable space with $\dim R \leqq n$, then from the above stated theorem there exists a sequence $\mathfrak{U}_1 > \mathfrak{U}_2^{**} > \mathfrak{U}_2 > \mathfrak{U}_3^{**} > \cdots$[5] of open coverings of R such that $S(p, \mathfrak{U}_m)$ $(m=1, 2, \cdots)$ is a nbd basis for each point p of R and such that each $S^2(p, \mathfrak{U}_{m+1}^*)$ intersects at most $n+1$ sets of $\mathfrak{U}_m$. Now, we define $S_{m_2, m_3, \cdots, m_p}(U)$ for $1 \leqq m_1 < m_2 < \cdots < m_p$ and for $U \in \mathfrak{U}_{m_1}$ such that $S_{m_2}(U) = \smile\{U' \mid S(U', \mathfrak{U}_{m_2}) \frown U \neq \phi,\ U' \in \mathfrak{U}_{m_2}\} = S^2(U, \mathfrak{U}_{m_2})$, $S_{m_2, \cdots, m_p}(U) = \smile\{U' \mid S(U', \mathfrak{U}_{m_p}) \frown S_{m_2, \cdots, m_{p-1}}(U) \neq \phi,\ U' \in \mathfrak{U}_{m_p}\} = S^2(S_{m_2, \cdots, m_{p-1}}(U), \mathfrak{U}_{m_p})$ and $S_{m_2, \cdots, m_p}(U) = U$ for $p=1$. Furthermore we define $\mathfrak{S}_{m_1} = \mathfrak{U}_{m_1}$,

1) The equivalence of these two definitions was proved by M. Katětov: On the dimension of non-separable spaces I, Czechoslovak Mathematical Journal, **2** (77) (1952), and by K. Morita: Normal families and dimension theory for metric spaces, Math. Annalen, **128** (1954), independently.

2) A theorem of dimension theory, Proc. Japan Acad., **32**, No. 3 (1956).

3) $S(p, \mathfrak{B}) = \smile\{V \mid p \in V \in \mathfrak{B}\}$, $S(A, \mathfrak{B}) = \smile\{V \mid V \in \mathfrak{B},\ A \frown V \neq \phi\}$ for a subset A and a covering $\mathfrak{B}$.

4) $S_{\varepsilon/2}(x) = \{y \mid \rho(x, y) < \varepsilon/2\}$, $\rho(S_{\varepsilon/2}(x), y_i) = \inf\{\rho(z, y_i) \mid z \in S_{\varepsilon/2}(x)\}$.

5) $\mathfrak{U}^* = \{S(U, \mathfrak{U}) \mid U \in \mathfrak{U}\}$, $\mathfrak{U}^{**} = (\mathfrak{U}^*)^*$, $S^2(p, \mathfrak{U}) = S(S(p, \mathfrak{U}), \mathfrak{U})$.

$\mathfrak{S}_{m_1,\cdots,m_p}=\{S_{m_2,\cdots,m_p}(U)|U\in\mathfrak{U}_{m_1}\}$.

Firstly, we prove that $\dfrac{1}{2^{m_1}}+\cdots+\dfrac{1}{2^{m_p}}\geqq\dfrac{1}{2^{l_1}}+\cdots+\dfrac{1}{2^{l_q}}$ implies $\mathfrak{S}_{m_1,\cdots,m_p}>\mathfrak{S}_{l_1,\cdots,l_q}$. If it holds $p\geqq q$ and $m_i=l_i$ $(i=1,\cdots,q)$, then the validity of $\mathfrak{S}_{m_1,\cdots,m_p}>\mathfrak{S}_{l_1,\cdots,l_q}$ is obvious. Next, we see easily that generally $S_{k_2,\cdots,k_r}(U')\subseteq S(U',\mathfrak{U}_{k_1})$ for $U'\in\mathfrak{U}_{k_1}$. For $S_{k_2,\cdots,k_r}(U')=S^2(S_{k_2,\cdots,k_{r-1}}(U'),\mathfrak{U}_{k_r})\subseteq S(S_{k_2,\cdots,k_{r-1}}(U'),\mathfrak{U}_{k_{r-1}})\subseteq S(S_{k_2,\cdots,k_{r-2}}(U'),\mathfrak{U}_{k_{r-2}})\subseteq\cdots\subseteq S(S_{k_2}(U'),\mathfrak{U}_{k_2})\subseteq S(U',\mathfrak{U}_{k_1})$ from $\mathfrak{U}_{k_1}>\mathfrak{U}_{k_2}^{**}>\mathfrak{U}_{k_2}>\mathfrak{U}_{k_3}^{**}>\cdots>\mathfrak{U}_{k_r}^{**}$. Hence, if $1/2^{m_1}+\cdots+1/2^{m_p}>1/2^{l_1}+\cdots+1/2^{l_q}$ and $m_1=l_1$, $m_2=l_2,\cdots$, $m_{i-1}=l_{i-1}$, $m_i<l_i$ for a definite i with $2\leqq i\leqq p$, q, then $S_{l_{i+1},\cdots,l_q}(U')\subseteq S(U',\mathfrak{U}_{l_i})\subseteq U''$ for every $U'\in\mathfrak{U}_{l_i}$ and for some $U''\in\mathfrak{U}_{m_i}$. If further U' satisfies $S(U',\mathfrak{U}_{l_i})\frown S_{l_2,\cdots,l_{i-1}}(U)\neq\phi$, then it holds $U''\frown S_{m_2,\cdots,m_{i-1}}(U)=U''\frown S_{l_2,\cdots,l_{i-1}}(U)\neq\phi$. Therefore $S_{l_{i+1},\cdots,l_q}(U')\subseteq U''\subseteq S_{m_2,\cdots,m_i}(U)\subseteq S_{m_2,\cdots,m_p}(U)$ for every U': $S_{l_2,\cdots,l_{i-1}}(U)\leftarrow U'\in\mathfrak{U}_{l_i}$, where we denote by $A\leftarrow U'\in\mathfrak{U}_l$ the relation that $S(U',\mathfrak{U}_l)\frown A\neq\phi$ for $U'\in\mathfrak{U}_l$ and shall use this notation from now forth. Hence $S_{l_2,\cdots,l_q}(U)=S_{l_2,\cdots,l_{i-1}}(U)\smile[\smile\{S_{l_{i+1},\cdots,l_q}(U')|S_{l_2,\cdots,l_{i-1}}(U)\leftarrow U'\in\mathfrak{U}_{l_i}\}]\subseteq S_{m_2,\cdots,m_p}(U)$. If $m_1<l_1$, then $S_{l_2,\cdots,l_q}(U')\subseteq S(U',\mathfrak{U}_{l_1})\subseteq U''\subseteq S_{m_2,\cdots,m_p}(U'')$ for every $U'\in\mathfrak{U}_{l_1}$ and for some $U''\in\mathfrak{U}_{m_1}$. Thus we obtain $\mathfrak{S}_{m_1,\cdots,m_p}>\mathfrak{S}_{l_1,\cdots,l_q}$.

2. Now we define a non-negative valued function $\rho(x,y)$ on $R\times R$ such that $\rho(x,y)=\inf\{1/2^{m_1}+\cdots+1/2^{m_p}|y\in S(x,\mathfrak{S}_{m_1,\cdots,m_p})\}$, $\rho(x,y)=1$ $y\notin S(x,\mathfrak{S}_{m_1,\cdots,m_p})$ for every m_i $(i=1,\cdots,p)$. Let us show that $\rho(x,y)$ satisfies the axiom of metric.

Since $S(p,\mathfrak{U}_m)$ $(m=1,2,\cdots)$ is a nbd basis of p, $\rho(x,y)$ obviously agrees with the topology of R, *i.e.* $S_\varepsilon(x)=\{y|\rho(x,y)<\varepsilon\}$ $(\varepsilon>0)$ is a nbd basis of each point p of R.

To prove the triangle axiom: $\rho(x,y)+\rho(y,z)\geqq\rho(x,z)$, we assume that $\rho(x,y)=a\geqq b=\rho(y,z)$. For an arbitrary $\varepsilon>0$ we can take m_1, $\cdots$, m_p, l_1, $\cdots$, l_q such that $1\leqq m_1<\cdots<m_p$, $1\leqq l_1<\cdots<l_q$, $a+\varepsilon>1/2^{m_1}+\cdots+1/2^{m_p}>a$, $b+\varepsilon>1/2^{l_1}+\cdots+1/2^{l_q}>b$ and such that $1/2^{m_1}+\cdots+1/2^{m_p}>1/2^{l_1}+\cdots+1/2^{l_q}$. Since $y\in S(x,\mathfrak{S}_{m_1,\cdots,m_p})$, $z\in S(y,\mathfrak{S}_{l_1,\cdots,l_q})$ are obvious from the definition of $\rho(x,y)$, we assume $x,y\in S_{m_2,\cdots,m_p}(U)$, $U\in\mathfrak{U}_{m_1}$; $y,z\in S_{l_2,\cdots,l_q}(V)$, $V\in\mathfrak{U}_{l_1}$. We notice that we can assume p, $q\geqq2$, $m_p>l_1$ without loss of generality.

i) Let us consider firstly the case of $m_1=l_1$. Since, as is firstly proved, $S_{m_2,\cdots,m_p}(U)S\subseteq(U,\mathfrak{U}_{m_1})$ and $S_{l_2,\cdots,l_q}(V)\subseteq S(V,\mathfrak{U}_{l_1})=S(V,\mathfrak{U}_{m_1})$, $x,z\in S(U,\mathfrak{U}_{m_1})\smile S(V,\mathfrak{U}_{m_1})\subseteq W$ for some $W\in\mathfrak{U}_{m_1-1}$ from $\mathfrak{U}_{m_1}^{**}<\mathfrak{U}_{m_1-1}$. Hence $z\in S(x,\mathfrak{S}_{m_1-1})$, and hence $\rho(x,z)\leqq1/2^{m_1-1}\leqq1/2^{m_1}+\cdots+1/2^{m_p}+1/2^{l_1}+\cdots+1/2^{l_q}=a+b-2\varepsilon$.

ii) To consider the case of $m_1<l_1$, we notice that there exist U_i $(i=1,\cdots,p)$, V_j $(j=1,\cdots,q)$ such that $U_i\in\mathfrak{U}_{m_i}$, $V_j\in\mathfrak{U}_{l_j}$, $y\in U_p\frown V_q$, $U=U_1\leftarrow U_2\leftarrow\cdots\leftarrow U_p$, $V=V_1\leftarrow V_2\leftarrow\cdots\leftarrow V_q$ and such that $x\in S_{m_2,\cdots,m_p}(U_1)$, $z\in S_{l_2,\cdots,l_q}(V_1)$. We take $i\geqq1$ such that $m_i<l_1\leqq m_{i+1}$.

a) In the case that $l_1 < m_{i+1}$ we can take $S_1, S_2 \in \mathfrak{U}^*_{l_1+1}$ such that $y \in S_1 \frown S_2$, $S_1 \frown U_i \neq \phi$, $S_2 \frown V_1 \neq \phi$ from the fact that $l_1 + 1 \leqq m_{i+1}$, $l_1 + 1 \leqq l_2$. For from $U_{i+1} \in \mathfrak{U}_{m_{i+1}} < \mathfrak{U}_{l_1+1}$ we get $y \in S_{m_{i+2},\cdots,m_p}(U_{i+1}) \subseteq S(U_{i+1}, \mathfrak{U}_{m_{i+1}}) \subseteq S(U_{i+1}, \mathfrak{U}_{l_1+1}) \subseteq S_1$ for some $S_1 \in \mathfrak{U}^*_{l_1+1}$ and from $V_2 \in \mathfrak{U}_{l_2} < \mathfrak{U}_{l_1+1}$ we get $y \in S_{l_3,\cdots,l_q}(V_2) \subseteq S(V_2, \mathfrak{U}_{l_2}) \subseteq S_2$ for some $S_2 \in \mathfrak{U}^*_{l_1+1}$. Then $S_1 \frown U_i \neq \phi$ and $S_2 \frown V_1 \neq \phi$ are obvious from the fact $U_i \leftarrow U_{i+1}$, $V_1 \leftarrow V_2$. Since $S_1 \frown S_2 \neq \phi$ and $\mathfrak{U}^{**}_{l_1+1} < \mathfrak{U}_{l_1}$, it holds $S_1 \smile S_2 \subseteq W$ for some $W \in \mathfrak{U}_{l_1}$. Hence $S(V_1, \mathfrak{U}_{l_1}) \frown U_i \neq \phi$, *i.e.* $U_i \leftarrow V_1 \in \mathfrak{U}_{l_1}$. Hence $z \in S_{l_1,\cdots,l_q}(U_i) \subseteq S_{m_2,\cdots,m_i,l_1,\cdots,l_q}(U_1)$. Since $x \in S(U'_{i+1}, \mathfrak{U}_{m_{i+1}}) \nsubseteq (U'_i)^c$ for some $U'_{i+1} \in \mathfrak{U}_{m_{i+1}}$, $U'_i \in \mathfrak{U}_{m_i}$, it holds $x \in S' \nsubseteq (U'_i)^c$ for some $S' \in \mathfrak{U}_{l_1}$. Therefore $S_{m_2,\cdots,m_i,l_1,\cdots,l_q}(U_1) \ni x, z$, and hence $z \in S(x, \mathfrak{S}_{m_1,\cdots,m_i,l_1,\cdots,l_q})$. Thus we get $\rho(x, y) \leqq 1/2^{m_1} + \cdots + 1/2^{m_i} + 1/2^{l_1} + \cdots + 1/2^{l_q} \leqq 1/2^{m_1} + \cdots + 1/2^{m_p} + 1/2^{l_1} + \cdots + 1/2^{l_q} = a + b + 2\varepsilon$.

b) If $m_{i+1} = l_1$, then we take k such that $0 \leqq k \leqq i$; $m_{i+1} - 1 = m_i$, $m_i - 1 = m_{i-1}$, $\cdots$, $m_{i-k+2} - 1 = m_{i-k+1}$, $m_{i-k+1} - 1 > m_{i-k}$, where $k = 0$ for $m_{i+1} - 1 > m_i$, and $k = i$ for $m_{j+1} - 1 = m_j$ $(j = 1, 2, \cdots, i)$. In the case of $k < i$, from $S(U_{i-k+1}, \mathfrak{U}_{m_{i-k+1}}) \frown U_{i-k} \neq \phi$ and $y \in S(U_{i-k+1}, \mathfrak{U}_{m_{i-k+1}}) \frown S(V_1, \mathfrak{U}_{l_1}) \neq \phi$ we get $W \in \mathfrak{U}_{m_{i-k+1}-1}$ such that $W \frown U_{i-k} \neq \phi$, $W \supseteq S(V_1, \mathfrak{U}_{l_1}) \ni z$. Therefore $z \in S_{m_2,\cdots,m_{i-k},m_{i-k+1}-1}(U_1)$. With respect to x, since $x \in S(U'_{i-k+1}, \mathfrak{U}_{m_{i-k+1}}) \nsubseteq (S_{m_2,\cdots,m_{i-k}}(U_1))^c$ for some $U'_{i-k+1} \in \mathfrak{U}_{m_{i-k+1}}$, there exists $W' \in \mathfrak{U}_{m_{i-k+1}-1}$ such that $x \in W' \nsubseteq (S_{m_2,\cdots,m_{i+k}}(U_1))^c$. Hence $x \in S_{m_2,\cdots,m_{i-k},m_{i-k+1}-1}(U_1)$. Thus we get $z \in S(x, \mathfrak{S}_{m_1,\cdots,m_{i-k},m_{i-k+1}-1})$, and hence $\rho(x, z) \leqq 1/2^{m_1} + \cdots + 1/2^{m_{i-k}} + 1/2^{m_{i-k+1}-1} = 1/2^{m_1} + \cdots + 1/2^{m_{i+1}} + 1/2^{l_1} \leqq a + b + 2\varepsilon$.

In the case of $i = k$, we get $W \in \mathfrak{U}_{m_1-1}$ such that $x \in W \supseteq S(V_1, \mathfrak{U}_{l_1})$. Hence $z \in S(x, \mathfrak{S}_{m_1-1})$, and consequently $\rho(x, z) \leqq 1/2^{m_1-1} = 1/2^{m_1} + \cdots + 1/2^{m_{i+1}} + 1/2^{l_1} \leqq a + b + 2\varepsilon$. Thus we get $\rho(x, z) \leqq a + b + 2\varepsilon$ for every $\varepsilon > 0$ in any case, and hence it must be $\rho(x, z) \leqq a + b = \rho(x, y) + \rho(y, z)$.

3. Now, it remains to prove that $\rho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ $(i = 1, \cdots, n+2)$ imply $\rho(y_i, y_j) < \varepsilon$ for some i, j with $i \neq j$. Since $\rho(S_{\varepsilon/2}(x), y_i) < \varepsilon$, we can choose $n+2$ points x_i and a positive number δ such that $\rho(x, x_i) < \delta < \varepsilon/2$, $\rho(x_i, y_i) < \varepsilon$. Then there exist $m_1, \cdots, m_p$ and $S_i \in \mathfrak{S}_{m_1,\cdots,m_p}$ $(i = 1, \cdots, n+2)$ such that $x_i, y_i \in S_i$, $2\delta < 1/2^{m_1} + \cdots + 1/2^{m_p} < \varepsilon$. Since $\delta < 1/2^{m_1+1} + \cdots + 1/2^{m_p+1}$, it must be $x_i \in S(x, \mathfrak{S}_{m_1+1,\cdots,m_p+1})$ $(i = 1, \cdots, n+2)$. Let $S_i = S_{m_2,\cdots,m_p}(U_i)$, $U_i \in \mathfrak{U}_{m_1}$, then there exist $S_i' \in \mathfrak{U}^*_m$ $(< \mathfrak{U}^*_{m_1+1})$ such that $S_i' \frown U_i \neq \phi$, $S_i' \ni x_i$. Hence $S_i' \frown S(x, \mathfrak{U}^*_{m_1+1}) \neq \phi$ $(i = 1, \cdots, n+2)$ from $\mathfrak{S}_{m_1+1,\cdots,m_p+1} < \mathfrak{U}^*_{m_1+1}$, and hence $S^2(x, \mathfrak{U}^*_{m_1+1}) \frown U_i \neq \phi$ $(i = 1, \cdots, n+2)$. Since from the first assumption $S^2(x, \mathfrak{U}^*_{m_1+1})$ intersects at most $n+1$ sets of $\mathfrak{U}_{m_1}$, it must be $U_i = U_j$ for some i, j with $i \neq j$. Then $y_j \in S(y_i, \mathfrak{S}_{m_1,\cdots,m_n})$, and hence we conclude $\rho(y_i, y_j)$ $1/2^{m_1} \leqq + \cdots + 1/2^{m_p} < \varepsilon$.

Sufficiency. We denote by $\rho(x, y)$ a metric satisfying the condition of this theorem. Then we denote by M_1 a maximal subset of

R such that $x, y \in M_1$ and $x \neq y$ imply $\rho(x, y) \geqq 1/2$. From the maximal property of M_1, $\mathfrak{S}_1' = \{S_{1/2}(x) | x \in M_1\}$ is obviously an open covering of R. From the properties of $\rho(x, y)$ and of M_1 it is also obvious that $S_{1/2^2}(x)$ intersects at most $n+1$ sets of $\mathfrak{S}_1'$ for every point x of R. Put $\mathfrak{S}_1 = \{\smile\{S_{1/2^3}(y) | y \in S_{1/2}(x)\} | y \in M_1\}$, then every $S_{1/2^3}(x)$ intersects at most $n+1$ sets of $\mathfrak{S}_1$. If we put $\mathfrak{U}_r = \{S_r(x) | x \in R\}$, then $\mathfrak{U}^*_{1/2^5} < \mathfrak{U}_{1/2^3} < \mathfrak{S}_1 < \mathfrak{U}_{1/2+1/2^3}$.

Next we denote by M_2 a maximal subset of R such that $x, y \in M_2$ and $x \neq y$ imply $\rho(x, y) \geqq 1/2^6$. $\mathfrak{S}_2' = \{S_{1/2^6}(x) | x \in M_2\}$ covers R, and $S_{1/2^7}(x)$ intersects at most $n+1$ sets of $\mathfrak{S}_2'$ in the same way. Hence every $S_{1/2^8}(x)$ intersects at most $n+1$ sets of $\mathfrak{S}_2 = \{\smile\{S_{1/2^8}(y) | y \in S_{1/2^6}(x)\} | x \in M_2\}$. Furthermore it holds obviously $\mathfrak{U}^*_{1/2^{10}} < \mathfrak{U}_{1/2^8} < \mathfrak{S}_2 < \mathfrak{U}_{1/2^6+1/2^8} < \mathfrak{U}_{1/2^5} < \mathfrak{U}^*_{1/2^5} < \mathfrak{S}_1$. By repeating such processes we get a sequence $\mathfrak{S}_1 > \mathfrak{S}_2^* > \mathfrak{S}_2 > \mathfrak{S}_3^* > \cdots$ of open coverings of R such that every set of $\mathfrak{S}_{m+1}$ intersects at most $n+1$ sets of $\mathfrak{S}_m$ and such that $\mathfrak{S}_m < \mathfrak{U}_{1/2^{1+(m-1)5}+1/2^{3+(m-1)5}}$ $(m=1, 2, \cdots)$. Hence we conclude $\dim R \leqq n$ from the firstly stated theorem by the author.

As is easily seen from the proof of this theorem, we can state this theorem in the following form.

Corollary 1. *In order that $\dim R \leqq n$ for a metrizable space R it is necessary and sufficient that one can assign a metric $\rho(x, y)$ agreeing with the topology of R such that for every $\varepsilon > 0$ and for some $\varphi(\varepsilon) > 0$, $\rho(S_{\varphi(\varepsilon)}(x), y_i) < \varepsilon$ $(i=1, \cdots, n+2)$ imply $\rho(y_i, y_j) < \varepsilon$ for some i, j with $i \neq j$.*

We can deduce the following proposition proved by J. de Groot[6] from our theorem for the special case of $n=0$.

Corollary 2. *A metrizable space R is 0-dimensional if and only if one can assign a metric which satisfies $\rho(x, z) \leqq max[\rho(x,y), \rho(y,z)]$.*

Proof. Let $\dim R=0$, then from our theorem we can assign a metric $\rho(x, y)$ such that $\rho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ $(i=1, 2)$ imply $\rho(y_1, y_2) < \varepsilon$. Hence if we assume $\rho(x, z) = \varepsilon > \max[\rho(x, y), \rho(y, z)]$ for some $x, y, z \in R$, then $\rho(S_{\varepsilon/2}(y), x) < \varepsilon$, $\rho(S_{\varepsilon/2}(y), z) < \varepsilon$ and $\rho(x, z) = \varepsilon$, which contradict the character of $\rho(x, y)$. Therefore it must be $\rho(x, z) \leqq \max[\rho(x,y), \rho(y, z)]$.

Conversely, let $\rho(x, y)$ be a metric such that $\rho(x, z) \leqq \max[\rho(x, y), \rho(y, z)]$, and let us assume that $\rho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ $(i=1, 2)$ Then there exist $x_1, x_2 \in S_{\varepsilon/2}(x)$ such that $\rho(x_i, y_i) < \varepsilon$ $(i=1, 2)$. Since $\rho(x_1, x_2) < \varepsilon$, we get $\rho(y_1, y_2) \leqq \max[\rho(y_1, x_1), \rho(x_1, y_2)] \leqq \max[\rho(x_1, y_1), \rho(x_1, x_2), \rho(x_2, y_2)] < \varepsilon$.

6) Non Archimedean metrics in topology (1954), to be published. See J. de Groot and H. de Vries: A note on non-Archimedean metrizations, Proceedings Koninkl. Nederl. Akademie van Wetenschappen, Series A, **58**, No. 2 (1955).