52. Contribution to the Theory of Semi-groups. II

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Any compact semi-group contains at least one idempotent. This theorem has been proved by some writers (cf. K. Iséki [2], Th. 3).

Let E be the set of all idempotents e_α of a given compact semi-group S, then E is non-empty.

If $e_\alpha e_\beta = e_\alpha$ for e_α, $e_\beta \in E$, we shall write $e_\alpha \leq e_\beta$. The order relation $\leq$ defines a quasi-order on E. If E is commutative, then E is a partial order set relative to the order.

In this Note, we shall first extend a result of S. Schwarz [3].

We shall first prove that

$$\mathfrak{N} = \bigcap_{e_\alpha \in E} Se_\alpha S$$

is non-empty. By the compactness of S, each $Se_\alpha S$ is closed. For any finite $e_{\alpha_1}, e_{\alpha_2}, \cdots, e_{\alpha_k}$, we have

$$e_{\alpha_1} \cdot e_{\alpha_2} \cdots e_{\alpha_k} \in Se_{\alpha_1}S \cdot Se_{\alpha_2}S \cdots Se_{\alpha_k}S$$
$$\subseteq Se_{\alpha_1}S \frown Se_{\alpha_2}S \frown \cdots \frown Se_{\alpha_k}S.$$

Therefore, $Se_{\alpha_1}S \frown Se_{\alpha_2}S \frown \cdots \frown Se_{\alpha_k}S$ is non-empty, and $\mathfrak{N}$ is non-empty. It is clear that $\mathfrak{N}$ is a closed two-sided ideal, and hence $\mathfrak{N}$ is a compact semi-group. For $a \in \mathfrak{N}$, SaS is a closed ideal of $\mathfrak{N}$. The compact semi-group SaS contains an idempotent e. Therefore $SeS \subseteq SaS \subseteq \mathfrak{N} \subseteq SeS$. Hence $SaS = SeS = \mathfrak{N}$, for any a and any idempotent e of $\mathfrak{N}$. $\mathfrak{N} = SaS$ is a closed minimal two-sided ideal.

Thus, this fact shows that *there is a closed minimal two-sided ideal in S.*

If S is a compact homogroup in the sense of G. Thierrin [4], then S contains a compact group and two-sided ideal m of S. Therefore, $\mathfrak{N} \subset m$. As any group does not contain proper ideal, $\mathfrak{N} = m$. Therefore, $\mathfrak{N}$ is a compact group. Hence $\mathfrak{N}$ contains only one idempotent e, which is the unit element of $\mathfrak{N}$. Let e' be an idempotent of S, then, by the definition of $\mathfrak{N}$, $\mathfrak{N} \subseteq \mathfrak{N}e'\mathfrak{N} \subseteq \mathfrak{N}$. Hence $ee'e \in \mathfrak{N}$. Since S is a homogroup, e is permutable with any element of S. Hence $ee'e = ee'$ and ee' is an idempotent. Therefore $ee' = e$ and this shows $e \leq e'$. So we can state the following

*Theorem 1. Any compact homogroup has a unique least idempotent.**)

*) Theorem 1 is proved without the assumption of compactness.

It is well known that any compact abelian semi-group is homogroup (see K. Iséki [1]).

Therefore, Theorem 1 implies the following

Theorem 2. Any compact abelian semi-group has a unique least idempotent.

Theorem 2 has been proved by S. Schwarz [3].

Let S be a topological semi-group, and $\chi(x)$ a *continuous* homomorphism of S into the multiplicative group of complex numbers of absolute value one: $\chi(a)\chi(b)=\chi(ab)$ and $|\chi(a)|=1$. Such a $\chi(x)$ is called a *character* of S.

Let e be an idempotent of S. Then

$$\chi(e)=\chi(e^2)=(\chi(e))^2$$

and hence

$$\chi(e)(\chi(e)-1)=0.$$

Therefore, we have $\chi(e)=1.$

Suppose that S is a topological homogroup. By a theorem of G. Thierrin [4], there is an idempotent e such that $\mathfrak{N}=\{xe \mid x \in S\}$ is a group and two-sided ideal of S. $\mathfrak{N}$ is called the group ideal of S. Let $n \in \mathfrak{N}$, then it is clear that $S=\bigcup_{n\in\mathfrak{N}} A_n$, where $A_n=\{x \mid xe=n\}$. Let $\chi(x)$ be a character of S, and $a_n \in A_n$, then

$$a_n e=n$$

and

$$\chi(a_n)\chi(e)=\chi(n),$$

hence, we have $\chi(a_n)=\chi(n)$ by $\chi(e)=1$. Therefore, $\chi(n)$ is a character of $\mathfrak{N}$. On the other hand, let $\chi(n)$ be a character of $\mathfrak{N}$. We shall define a character $\psi(x)$ of S by $\chi(n)$. For $a \in S$, we define

$$\psi(a)=\chi(ae), \qquad ae \in \mathfrak{N}.$$

Then it is clear that $|\psi(a)|=1$ and

$$\psi(a)\psi(b)=\chi(ae)\chi(be)=\chi(aebe)=\chi(abe)$$
$$=\psi(ab).$$

To prove the continuity of $\psi(a)$, let ε be any positive number. Since $\chi(x)$ is continuous on $\mathfrak{N}$, we can find a neighbourhood $U(ae)$ such that $U(ae)\frown\mathfrak{N}\ni x$ implies

$$|\chi(ae)-\chi(x)|<\varepsilon.$$

For a neighbourhood $U(a)$ such that $U(a)e\subset U(ae)$, $b \in U(a)$ implies

$$be \in U(a)e \subset U(ae),$$

and $be \in \mathfrak{N}$. Hence $be \in U(ae)\frown\mathfrak{N}$ and

$$|\psi(a)-\psi(b)|=|\chi(ae)-\chi(be)|<\varepsilon.$$

Therefore $\psi(a)$ is continuous on S.

Let $\hat{S}$ denote the set of all characters of S. For χ, ψ of $\hat{S}$, the product $\chi\psi$ is defined as $\chi\psi(x)=\chi(x)\psi(x)$ for all $x \in S$. Then

$\hat{S}$ is a group. The result mentioned above shows that $\hat{S}$ is isomorphic to the group of characters of $\mathfrak{N}$. Therefore, we have the following

Theorem 3. The set of all characters of a topological homogroup is isomorphic to the group of characters of group ideal of it.

Theorem 2 is a generalisation of a result by S. Schwarz [3]: *the set of all characters of a compact abelian semi-group is isomorphic to the group of characters of maximal subgroup of it.*

References

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