77. On $H_*(\Omega^N(S^n); Z_2)$

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§ 1. In this note we shall give a brief account about the determination of the modulo 2 Pontrjagin ring $H_*(\Omega^N(S^n); Z_2)$ of the N-times iterated loop space $\Omega^N(S^n)$ of the n-sphere S^n, where $0<N<n$.

For this purpose we first introduce a new concept of an H_n-space, of which the $(n+1)$-times iterated loop space of a metrizable space is a typical example. Then we define some homological operation modulo 2, which may, at least formally, be regarded as dual to the Steenrod's squaring operations. In fact, although they are defined only within the category of H_n-spaces, necessary transgression theorems in the homology theory may be established with respect to these operations.

The complete discussions of our note will be published in a forthcoming Memoirs of the Faculty of Science, Kyusyu University.

§ 2. **H_n-spaces. Definition 1.** We say that a space X has an *H_n-structure* (or is an *H_n-space*), when there exists a system of maps $\{\theta_m\}$, $(0\leqq m\leqq n)$ subject to the following conditions:

(i) θ_m is a map

$$\theta_m: I^m\times X\times X\to X \tag{1.a$_m$}$$

where I is a unit interval and I^m the m-fold product of I; in particular

$$\theta_0: X\times X\to X. \tag{1.a$_0$}$$

(ii) θ_m's satisfy

$$\begin{aligned}\theta_m(t_1,\cdots,t_{i-1},0,t_{i+1},\cdots,t_m;x,y)&=\theta_{i-1}(t_1,\cdots,t_{i-1};x,y),\\ \theta_m(t_1,\cdots,t_{i-1},1,t_{i+1},\cdots,t_m;x,y)&=\theta_{i-1}(1-t_1,\cdots,1-t_{i-1};y,x),\end{aligned} \tag{1.b}$$

for any m, i, $(t_1,\cdots,t_m)\in I^m$, and $x,y\in X$.

(iii) There exists an element $e\in X$, called the *unit* of this H_n-structure, satisfying

$$\theta_m(t_1,\cdots,t_m;x,e)=\theta_m(t_1,\cdots,t_m;e,x)=x \tag{1.c}$$

for any $(t_1,\cdots,t_m)\in I^m$ and $x\in X$.

For an H_n-space X, θ_0 defines a product on X, and we may consider X as an H-space in the widest sense. It is called *homotopy-associative* if it is so when regarded as an H-space.

Let X be an H_n-space and $X_1=\Omega(X)$ the space of loops in X with e as the reference point, and X_1' be the space of paths ending at e, with the usual topology.

Proposition 1. *If X is an H_n-space, then X_1 and X_1' are also H_n-spaces.*

In fact, we define

(2) $\theta_m : I^m \times X_1 \times X_1 \to X_1$ (or $I^m \times X_1' \times X_1' \to X_1'$)

for $0 \leqq m \leqq n$ as follows:

(2′) $\theta_m(t_1, \cdots, t_m; \xi, \eta)(t) = \theta_m(t_1, \cdots, t_m; \xi(t), \eta(t))$

for any $\xi, \eta \in X_1$ (or $\in X_1'$), where t is the parameter of loops (or of paths).

These θ_m's obviously define an H_n-structure on X_1 (or on X_1') which we shall call the *induced* H_n-structure. We note that X_1 is a subspace of X_1', and the maps $\{\theta_m\}$ defined above for X_1 are the restrictions of those for X_1'.

Theorem 1. *Let X be a metric H_n-spaces and X_1, X_1' be as above. $\{\theta_m\}$, $0 \leqq m \leqq n$, will denote the maps for X as well as the induced ones in X_1 and X_1'. Then we may construct a map*

(3) $\theta_{n+1} : I^{n+1} \times X_1' \times X_1 \to X_1'$,

of the same properties as the other maps $\{\theta_m\}$, $0 \leq m \leq n$. Moreover

$$\theta_{n+1}(I^{n+1} \times X \times X) \subset X,$$

i.e., θ_{n+1}, together with induced θ_m's, defines an H_{n+1}-structure on X_1.

§ 3. **H-squaring operations.** For an H_n-space X we define operations[1] of degree i

(4) $\Theta_i : Q^N(X) \otimes Q^N(X) \to Q^N(X)$,

for $0 \leqq i \leqq n$, satisfying

(4′) $\omega\Theta_i(c_p \otimes c_q) = (-1)^{pq}\Theta_{i-1}(c_q \otimes c_p) + (-1)^i\Theta_{i-1}(c_p \otimes c_q)$,

where $c_p \in Q_p^N(X)$, $c_q \in Q_q^N(X)$, and ω is the boundary operator in the operator-complex.[1] For the definition of these operations we use essentially the maps θ_i, $0 \leqq i \leqq n$, with some auxiliary tools from the theory of operator-complexes. These operations enable us to define the following maps

(5) $Q_i : H_q(X; Z_2) \to H_{2q+i}(X; Z_2)$

for all q and i such that $q \geqq 0$ and $0 \leqq i \leqq n$. We call these Q_i, $0 \leqq i \leqq n$, *H-squaring operations* on an H_n-space X. $Q_0, Q_1, \cdots, Q_{n-1}$ are homomorphisms, while Q_n is not necessarily so. The basic property of our H-squaring operations is

Theorem 2. *Under the same hypotheses as in Theorem 1, we consider the fibering (X_1', X, π, X_1). Let τ denote the homology transgression homomorphism of this fibering. Then*

(6) $Q_{i+1} \circ \tau = \tau \circ Q_i$

for $0 \leqq i \leqq n$.

More precisely, if $u \in H_q(X; Z_2)$ is transgressive, then $Q_i(u)$, $0 \leqq i \leqq n$, is also transgressive; operation $Q_{m+1} : M_{q-1} \to M_{2q+m-1}$ is well defined for $0 \leqq m \leqq n$, where $M_q = H_q(X_1; Z_2)/\partial_* \circ p_*^{-1}(0)$; and there holds the commutativity (6).

1) Cf. N. Steenrod: Reduced powers of cohomology classes, Ann. Math., **56**, 47–67 (1952).

§4. **Proposition 2.** *Let X be a simply-connected metric homotopy-associative H_0-space and let X_1, X_1' be as above. If the Pontrjagin ring $H_*(X; Z_2)$ has a simple system of generators*[2] $\{x_1, x_2, \cdots, x_i, \cdots\}$ *consisting of transgressive elements x_i (with respect to the fibering (X_1', X, π, X_1)), then $H_*(X_1; Z_2)$ is a polynomial ring with generators $\{y_1, y_2, \cdots, y_i, \cdots\}$, where y_i is a transgression image of x_i.*

For the proof of this proposition we need a lemma[3] due to J.-P. Serre about an algebraic property of spectral sequences.

Let X be a metric space with metric ρ, then $\Omega(X)$ is also a metric space; obviously the topologies of $\Omega(X)$ (defined by this metric and by the compact-open topology) are the same. $\Omega(X)$ is an H-space with respect to the usual composition of loops. We give another H-structure on $\Omega(X)$ by the following rule of composition

$$(7)\qquad (l_1 \square l_2)(t) = \begin{cases} l_1\left(\dfrac{t}{\alpha}\right) & \text{for } 0 \leqq t \leqq \alpha, \\ l_2\left(\dfrac{t-\alpha}{1-\alpha}\right) & \text{for } \alpha \leqq t \leqq 1, \end{cases}$$

for any $l_1, l_2 \in \Omega(X)$, where $\alpha = \dfrac{\rho(e, l_1)}{\rho(e, l_1)+\rho(e, l_2)}$. This H-structure is homotopy-associative and an H_0-structure in our sense. Since the identity map of $\Omega(X)$ onto itself is easily seen to be an H-equivalence between the above two H-structures, the Pontrjagin rings $H_*(\Omega(X); Z_2)$ are the same for these two H-structures. Consequently, without loss of generality we may consider $\Omega(X)$ as a metric homotopy-associative H_0-space if X is a metric space.

After these preparations we can determine the Pontrjagin ring $H_*(\Omega^N(S^n); Z_2)$ $(N<n)$, by making use of Theorem 2 and Proposition 2, in entirely the same way as the determination of $H^*(\Pi, n; Z_2)$.[4] That is, if we use the symbol Q^i instead of Q_i and denote $Q^{i_1}Q^{i_2}\cdots Q^{i_r}$ by Q^J, $J=\{i_1, i_2, \cdots, i_r\}$, we have

Theorem 3. *Pontrjagin ring $H_*(\Omega^N(S^n); Z_2)$ $(N<n)$ is a polynomial ring whose generators are u_{n-N} and all admissible $Q^J u_{n-N}$'s such that $\epsilon(J)>n-N$ and $l(J)<n$, where u_{n-N} is an element of degree $n-N$ and $J=\{i_1, \cdots, i_r\}$ is called admissible when $i_{s-1} \leqq 2i_s$ for $2 \leqq s \leqq r$, and $\epsilon(J)=i_1-i_2-\cdots-i_r$, $l(J)=i_r$.*

2) This notion is due to A. Borel. Cf. A. Borel: Sur la cohomologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compactes, Ann. Math., **57**, 115–207 (1953).

3) This lemma was suggested by Serre on the occasion of his visiting our Institute. It asserts: "Let $E_r^{a,b}$, $E_r'^{a,b}$, $r \geqq 2$, be two algebraic homological (or cohomological) spectral sequences bigraded with non-negative degrees, over a field k, and $E_2^{a,b}=E_2^{a,0} \otimes E_2^{0,b}$, $E_2'^{a,b}=E_2'^{a,0} \otimes E_2'^{0,b}$. Moreover, let $\{h_r, r \geqq 2\}$ be a homomorphism from $\{E_r\}$ to $\{E_r'\}$. Then, in the following three assertions, (α) $h_2^{a,0}$ is isomorphic for all $a \geqq 0$, (β) $h_2^{0,b}$ is isomorphic for all $b \geqq 0$, (γ) $h_2^{a,b}$ is isomorphic for all $a, b \geqq 0$, any two conclude the rest".

4) Cf. J.-P. Serre: Cohomologie mod 2 des complexes d'Eilenberg-MacLane, Comment. Math. Helvet., **27**, 198–232 (1953).