89. On Closed Mappings. II

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(Comm. by K. KUNUGI, M.J.A., June 12, 1956)

The present note is a continuation of our previous paper on the closed mappings.[1] Let S and E be T_1-spaces. A mapping from S onto E is said to be closed if the image of every closed subset of S is closed in E. Recently it has been shown that several topological properties are invariant under a closed continuous mapping under some restrictions.[2]

In this note, we will prove the invariance of other topological properties under a closed continuous mapping and under the inverse mapping of it, under some restrictions.

1. Let us recall some definitions in the following. The space S is called paracompact (point-wise paracompact) if every open covering of S has an open locally finite (point-finite) refinement and countably paracompact if every countable open covering has an open locally finite refinement. The space S is said to have the star-finite property if every open covering of S has an open star-finite refinement. By an S-space, we mean a normal space with the star-finite property according to E. G. Begle.[3]

Theorem 1. *Let f be a closed continuous mapping from a normal space S onto a normal space E. If the inverse image $f^{-1}(p)$ is compact for every point p of E, then the countable paracompactness is invariant under f.*

Proof. Since f is a closed continuous mapping, the image space E is normal by a theorem of G. T. Whyburn.[4] Let $\{F_i\}$ be a decreasing sequence of closed sets in E with vacuous intersection. Then $\{f^{-1}(F_i)\}$ is a decreasing sequence of closed sets in S with vacuous intersection since f is continuous. Since S is countably paracompact and normal, there exists a sequence $\{G_i\}$ of open sets such that $\bigcap_{i=1}^{\infty} G_i=\phi$ and $f^{-1}(F_i)\subset G_i$ $(i=1, 2,\cdots)$.[5] Since f is closed and continu-

1) S. Hanai: On closed mappings, Proc. Japan Acad., **30**, 285–288 (1954).

2) G. T. Whyburn: Open and closed mappings, Duke Math. Jour., **17**, 69–74 (1950). A. V. Martin: Decompositions and quasi-compact mappings, Duke Math. Jour., **21**, 463–469 (1954). V. K. Balachandran: A mapping theorem for metric spaces, Duke Math. Jour., **22**, 461–464 (1955). K. Morita and S. Hanai: Closed mappings and metric spaces, Proc. Japan Acad., **32**, 10–14 (1956).

3) E. G. Begle: A note on S-spaces, Bull. Amer. Math. Soc., **55**, 577–579 (1949).

4) G. T. Whyburn: Loc. cit.

5) C. H. Dowker: On countably paracompact spaces, Canadian Jour. Math., **3**, 219–224 (1951).

ous, each $(G_i)_0$ is an open inverse set and $f^{-1}(F_i)\subset(G_i)_0\subset G_i$ and $\bigcap_{i=1}^{\infty}(G_i)_0=\phi$ where $(G_i)_0$ denotes the union of all $f^{-1}(p)$ such that $f^{-1}(p)\subset G_i$. Then it is obvious that $F_i\subset f\{(G_i)_0\}$ $(i=1, 2,\cdots)$, $\bigcap_{i=1}^{\infty} f\{(G_i)_0\}=\phi$ and each $f\{(G_i)_0\}$ is open since f is a closed continuous mapping. Hence, by C. H. Dowker's theorem,[6] E is countably paracompact. This completes the proof.

Theorem 2. *Let f be a closed continuous mapping from a normal space S onto a normal space E such that the inverse image $f^{-1}(p)$ is compact for every point p of E. If S is a locally compact S-space, then so is E.*

Proof. Since S is an S-space, S is paracompact and normal. Hence E is paracompact and normal since f is a closed continuous mapping such that $f^{-1}(p)$ is compact for every point p of E.[7] Let $\mathfrak{M}=\{M_\alpha\}$ be an open covering of E, then $\mathfrak{M}$ has an open locally finite refinement $\mathfrak{N}=\{N_\beta\}$. Then $\mathfrak{N}'=\{f^{-1}(N_\beta)\}$ is an open covering of S since f is continuous. Since S is an S-space, $\mathfrak{N}'$ has an open star-finite refinement $\mathfrak{R}'=\{R'_\gamma\}$. Since S is locally compact and f is a closed continuous mapping such that $f^{-1}(p)$ is compact for every point p of E, E is locally compact.[8]

For each point p of E, we can find an open neighborhood $O(p)$ of p such that $\overline{O(p)}$ is compact and intersects only a finite number of sets of $\mathfrak{N}$. Then $\mathfrak{K}=\{O(p)\mid p\in E\}$ is an open covering of E. Since E is paracompact, $\mathfrak{K}$ has an open locally finite refinement $\mathfrak{G}=\{G_\delta\}$. Then each set $\overline{G}_\delta$ is compact and intersects only a finite number of sets of $\mathfrak{N}$. Since each $f^{-1}(p)$ is compact, there exists a finite number of sets of $\mathfrak{R}'$ which covers $f^{-1}(p)$, say $\{R_i^{(p)\prime}\}$ $(i=1, 2,\cdots, n(p))$.

Let $G_{\delta(p)}$ be a set of $\mathfrak{G}$ containing p and let $\{N_j^{(p)}\}$, $(j=1, 2,\cdots, k(p))$, be the set of all sets of $\mathfrak{N}$ intersecting $G_{\delta(p)}$. Then the family of open sets $\{(\sum_{i=1}^{n(p)} R_i^{(p)\prime})_0\cap f^{-1}(G_{\delta(p)})\cap f^{-1}(N_j^{(p)}),\ j=1, 2, \cdots, k(p)\mid p\in E\}$ is evidently an open covering of S. Let $\mathfrak{R}=\{f\{(\sum_{i=1}^{n(p)} R_i^{(p)\prime})_0\}\cap G_{\delta(p)}\cap N_j^{(p)},\ j=1, 2,\cdots, k(p)\mid p\in E\}$, then $\mathfrak{R}$ is an open refinement of $\mathfrak{M}$ since $\mathfrak{R}$ is an open refinement of $\mathfrak{N}$. Let $R_j^{(p)}=f\{(\sum_{i=1}^{n(p)} R_i^{(p)\prime})_0\}\cap G_{\delta(p)}\cap N_j^{(p)}$, then $\mathfrak{R}=\{R_j^{(p)},\ j=1, 2,\cdots, k(p)\mid p\in E\}$.

We will next prove that $\mathfrak{R}$ has the star-finite property.

Suppose on the contrary that there exists a set $R_j^{(p)}$ which intersects infinitely many sets of $\mathfrak{R}$, say $\{R_{j(l)}^{(p_l)}\}$, $(l=1, 2,\cdots)$. Then

6) C. H. Dowker: Loc. cit.

7) K. Morita and S. Hanai: Loc. cit.

8) S. Hanai: Loc. cit.

$R_j^{(p)} \cap R_{j(l)}^{(p_l)} \neq \phi$ $(l=1, 2, \cdots)$. Hence

$$(*) \qquad f\{(\sum_{i=1}^{n(p)} R_i^{(p)\prime})_0\} \cap G_{\delta(p)} \cap N_j^{(p)} \cap f\{(\sum_{i=1}^{n(p_l)} R_i^{(p_l)\prime})_0\} \cap G_{\delta(p_l)} \cap N_{j(l)}^{(p_l)} \neq \phi,$$

$(l=1, 2, \cdots)$.

Since $\mathfrak{G}$ is an open locally finite covering and $\overline{G}_{\delta(p)}$ is compact and intersects only a finite number of $\mathfrak{N}$, the sequence $\{N_{j(l)}^{(p_l)}\}$ $(l=1, 2, \cdots)$ contains only a finite number of sets and $\overline{G}_{\delta(p)}$ intersects only a finite number of sets of $\mathfrak{G}$. Hence, from $(*)$, we can find a set $R_i^{(p)\prime}$ which intersects infinitely many $R_i^{(p_l)\prime}$. This contradicts that $\mathfrak{R}'$ is an open star-finite covering. This completes the proof.

Remark. In the above two theorems, the condition that the inverse image $f^{-1}(p)$ is compact for every point p of E can be replaced by that the boundary of $f^{-1}(p)$ is compact for every point p of E.[9]

2. In this section, we will deal with the case of the inverse mapping of a closed continuous mapping.

Theorem 3. *Let f be a closed continuous mapping from a normal space S onto a normal space E. If the inverse image $f^{-1}(p)$ is compact for every point p of E, then the countable paracompactness is invariant under the inverse mapping of f.*

Proof. As the proof of Theorem 1, we will prove this theorem by use of C. H. Dowker's theorem. Let $\{F_i\}$ $(i=1, 2, \cdots)$ be a decreasing sequence of closed sets in S such that $\bigcap_{i=1}^{\infty} F_i=\phi$. Then it is easy to see that $\lim_{i=\infty} F_i=\bigcap_{i=1}^{\infty} F_i=\phi$. Then we have $\lim_{i=\infty} f(F_i)=\phi$.

In fact, let q be any point of E and let x be any point of $f^{-1}(q)$, then we can find an open neighborhood $O(x)$ of x which intersects only a finite number of F_i since $\lim_{i=\infty} F_i=\phi$. If we take such $O(x)$ for each point x of $f^{-1}(q)$, we have the collection $\{O(x)\}$ which covers $f^{-1}(q)$. Since $f^{-1}(q)$ is compact, we can find a finite subcovering $\{O(x_i)\}$ $(i=1, 2, \cdots, n)$ of $\{O(x)\}$. Then $(\sum_{i=1}^{n} O(x_i))_0$ is an open inverse set since f is a closed continuous mapping. Then $f\{(\sum_{i=1}^{n} O(x_i))_0\}$ is an open neighborhood of q and intersects only a finite number of $\{f(F_i)\}$. Hence $q \in \limsup_{i=\infty} (F_i)$. Therefore we have $\lim_{i=\infty} f(F_i)=\phi$. Then we get $\bigcap_{i=1}^{\infty} f(F_i)=\phi$ from that $\lim_{i=\infty} f(F_i)=\phi$.

Since E is countably paracompact and normal, there exists a sequence $\{H_i\}$ of open sets in E such that $f(F_i) \subset H_i$ $(i=1, 2, \cdots)$ and $\bigcap_{i=1}^{\infty} H_i=\phi$. Hence $\bigcap_{i=1}^{\infty} f^{-1}(H_i)=\phi$ and $F_i \subset f^{-1}(H_i)$ where each $f^{-1}(H_i)$ is open since f is continuous. Therefore S is countably paracompact. This completes the proof.

9) K. Morita and S. Hanai: Loc. cit.

Theorem 4. *Let f be a closed continuous mapping from a normal space S onto a normal space E. If the inverse image $f^{-1}(p)$ is compact for every point p of E, then the paracompactness (point-wise paracompactness) is invariant under the inverse mapping of f.*

Proof. As the proof of the invariance of the point-wise paracompactness can be carried out in the similar way as that of the paracompactness, we will only prove the case for the paracompactness in the following.

Let $\mathfrak{M}=\{M_\alpha\}$ be an open covering of S. Since $f^{-1}(p)$ is compact for every point p of E, there exists a finite subcollection $\{M_i^{(p)}\}$ $(i=1, 2,\cdots, n(p))$ of $\mathfrak{M}$ such that $f^{-1}(p)\subset\sum_{i=1}^{n(p)} M_i^{(p)}$. We take such a finite subcollection $\{M_i^{(p)}\}$ of $\mathfrak{M}$ corresponding to each point p, and let $\mathfrak{M}'$ be the collection of all $M_i^{(p)}$ of such $\{M_i^{(p)}\}$ (p ranging over all points of E). Then $\mathfrak{M}'$ is an open refinement of $\mathfrak{M}$. Let $M(p)=(\sum_{i=1}^{n(p)} M_i^{(p)})_0$, then $M(p)$ is an open inverse set since f is a closed continuous mapping. Let $H(p)=f\{M(p)\}$, then $H(p)$ is an open set containing p. Then $\mathfrak{R}=\{H(p)\mid p\in E\}$ is an open covering of E. Since E is paracompact, $\mathfrak{R}$ has an open locally finite refinement $\mathfrak{R}'$. Then for each $R'\in\mathfrak{R}'$, we can find a point p such that $R'\subset H(p)$. Hence $f^{-1}(R')\subset M(p)\subset(\sum_{i=1}^{n(p)} M_i^{(p)})_0$. Then we have a collection $\mathfrak{N}=\{f^{-1}(R')\cap M_i^{(p)},\ i=1, 2,\cdots, n(p)\mid R'\in\mathfrak{R}'\}$ of open sets in S. It is evident that $\mathfrak{N}$ is an open refinement of $\mathfrak{M}'$. We will next prove that $\mathfrak{N}$ is locally finite.

Let x be any point of S and let $q=f(x)$, then there exists an open neighborhood $O(q)$ of q which intersects only a finite number of sets of $\mathfrak{R}'$, say $\{\mathfrak{R}_i'\}$ $(i=1, 2,\cdots, l)$, because $\mathfrak{R}'$ is locally finite. Then $f^{-1}\{O(q)\}\cap f^{-1}(R_i')\neq\phi$ $(i=1, 2,\cdots, l)$. By the definition of $\mathfrak{N}$, we can easily see that $f^{-1}\{O(q)\}$ intersects only a finite number of sets of $\mathfrak{N}$. Hence $\mathfrak{N}$ is locally finite. Therefore $\mathfrak{N}$ is an open locally finite refinement of $\mathfrak{M}$. This completes the proof.

Theorem 5. *Let f be a closed continuous mapping from a T_1-space S onto a T_1-space E. If the inverse image $f^{-1}(p)$ is compact for every point p of E, then the star-finite property is invariant under the inverse mapping of f.*

As we can prove this theorem in the similar way as Theorem 4, we omit the proof.

Since a T_2-space with the star-finite property is normal, we get easily the following corollary by virtue of Theorem 5.

Corollary. *Let f be a closed continuous mapping from a T_2-space S onto a T_2-space E such that the inverse image $f^{-1}(p)$ is compact for every point p of E. If E is an S-space, then so is S.*