103. *Fourier Series. I. Parseval Relation*

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1. In our former paper [1] we have proved the following theorems:

Theorem I. *If $g(t)$ is integrable and if $f(t)$ is a bounded measurable function such that*

$$\int_0^h (f(x+u)-f(x-u))\,du=o\left(h\Big/\log\frac{1}{h}\right)$$

uniformly for all x as $h\to 0$, then the Parseval relation

$$\frac{1}{\pi}\int_0^{2\pi} f(x)g(x)\,dx=\frac{a_0a_0'}{4}+\sum_{n=1}^{\infty}(a_na_n'+b_nb_n') \tag{1}$$

holds, where a_n, b_n and a_n', b_n' are Fourier coefficients of $f(t)$ and $g(t)$, respectively, and the right side series converges.

Theorem II. *If $g(t)$ is bounded measurable and if $f(t)$ is an integrable function such that*

$$\int_0^{2\pi} dx\left|\frac{1}{\delta}\int_0^{\delta}(f(x+u)-f(x-u))\,du\right|=o\left(1\Big/\log\frac{1}{\delta}\right),$$

then the Parseval relation (1) holds, where the right side series converges.

It is easy to see that if $f(t)$ and $g(t)$ belong to the conjugate classes and the Fourier series of one of them converges uniformly, then the Parseval relation (1) holds and the right side series converges in the ordinary sense.

Concerning the uniform convergence of Fourier series, the Salem theorem [3] (cf. [2]) is well known which reads as follows:

Theorem III. *If $f(t)$ is continuous and*

$$\sum_{k=1}^{[N/2]}\frac{f(t\pm 2k\pi/N)-f(t\pm(2k-1)\pi/N)}{k}$$

tends to zero uniformly as $N\to\infty$, then the Fourier series of $f(t)$ converges uniformly.

In this paper we show that if $f(t)$ and $g(t)$ belong to the conjugate classes, then the weaker one than the condition of Theorem III is sufficient for the validity of the Parseval relation (1).

2. **Theorem 1.** *If $g(t)$ is Lebesgue integrable and $f(t)$ is a bounded measurable function such that*

$$N\int_x^{x+\pi/N} dt\left|\sum_{k=1}^{[N/2]}\frac{f(t\pm(2k-1)\pi/N)-f(t\pm 2k\pi/N)}{k}\right| \tag{2}$$

is bounded uniformly in x and tends to zero almost everywhere, not necessarily uniform in x, then the Parseval relation (1) holds, where the right side converges.

Proof. Let S_N be the Nth partial sum of the series in (1), and $s_N(t, f)$ the Nth partial sum of the Fourier series of $f(t)$. Then

$$s_N(t, f)=\frac{1}{\pi}\int_0^{2\pi} f(x)D_N(x-t)dx,$$

where $D_N(t)$ is the Dirichlet kernel, that is,

$$D_N(t)=\frac{\sin(N+1/2)t}{2\sin t/2}.$$

Then

$$\begin{aligned}S_N&=\frac{a_0a_0'}{4}+\sum_{n=1}^{N}(a_na_n'+b_nb_n')=\frac{1}{\pi}\int_0^{2\pi} g(t)s_N(t, f)\,dt\\&=\frac{1}{\pi^2}\int_0^{2\pi}g(t)\,dt\int_{-\pi}^{\pi}f(x+t)D_N(x)\,dx\\&=\frac{1}{\pi}\int_{-\pi}^{\pi}D_N(x)\,dx\cdot\frac{1}{\pi}\int_0^{2\pi}g(t)f(x+t)\,dt.\end{aligned}$$

The right side is the Nth partial sum of the Fourier series of the function

$$h(x)=\frac{1}{\pi}\int_0^{2\pi}g(t)f(x-t)\,dt$$

at $x=0$. We shall now consider

$$\begin{aligned}\pi^2\Big(S_N-\frac{1}{\pi}\int_0^{2\pi}g(t)f(t)\,dt\Big)&=\int_{-\pi}^{\pi}\frac{\sin Nx}{x}dx\int_0^{2\pi}g(t)[f(x+t)-f(t)]dt+o(1)\\&=\int_0^{\pi}dx\int_0^{2\pi}dt+\int_{-\pi}^{0}dx\int_0^{2\pi}dt+o(1).\end{aligned}$$

Let us prove that the first integral is $o(1)$, the second may be similarly estimated, and hence we omit it. Let us write the first integral as follows:

$$\int_0^{\pi/N}dx\int_0^{2\pi}dt+\int_{\pi/N}^{\pi}dx\int_0^{2\pi}dt+o(1)=I+J+o(1).$$

We have

$$\begin{aligned}|I|&\leqq N\int_0^{\pi/N}dx\int_0^{2\pi}|g(t)|\,|f(x+t)-f(t)|\,dt\\&=\int_0^{2\pi}|g(t)|\,dt\cdot N\int_0^{\pi/N}|f(x+t)-f(t)|\,dx\end{aligned}$$

which is $o(1)$ as $N\to\infty$ since $g\in L(0, 2\pi)$ and the inner integral is uniformly $O(1/N)$ and is almost everywhere $o(1/N)$. We have also, supposing N as an odd integer,

$$J=\int_{\pi/N}^{\pi}\frac{\sin Nx}{x}dx\int_0^{2\pi}g(t)[f(x+t)-f(t)]dt$$
$$=\sum_{k=1}^{N-1}\int_{k\pi/N}^{(k+1)\pi/N}\frac{\sin Nx}{x}dx\int_0^{2\pi}g(t)[f(x+t)-f(t)]dt$$
$$=\sum_{k=1}^{(N-1)/2}\int_{\pi/N}^{2\pi/N}\sin Nx\,dx\left\{\frac{1}{x+(2k-1)\pi/N}\int_0^{2\pi}g(t)\left[f\left(x+\frac{(2k-1)\pi}{N}+t\right)\right.\right.$$
$$\left.\left.-f(t)\right]dt-\frac{1}{x+2k\pi/N}\int_0^{2\pi}g(t)\left[f\left(x+\frac{2k\pi}{N}+t\right)-f(t)\right]dt\right\}$$
$$=\sum_{k=1}^{(N-1)/2}\int_{\pi/N}^{2\pi/N}\sin Nx\,dx\left\{\frac{1}{(2k-1)\pi/N}\int_0^{2\pi}g(t)\left[f\left(x+\frac{(2k-1)\pi}{N}+t\right)\right.\right.$$
$$\left.-f\left(x+\frac{2k\pi}{N}+t\right)\right]dt$$
$$+\frac{x+\pi/N}{((2k-1)\pi/N)(x+2k\pi/N)}\int_0^{2\pi}g(t)\left[f\left(x+\frac{2k\pi}{N}+t\right)-f(t)\right]dt$$
$$-\frac{x}{((2k-1)\pi/N)(x+(2k-1)\pi/N)}\int_0^{2\pi}g(t)\left[f\left(x+\frac{(2k-1)\pi}{N}+t\right)\right.$$
$$\left.\left.-f(t)\right]dt\right\}$$
$$=J_1+J_2-J_3,$$

say, where

$$J_1=\int_0^{2\pi}g(t)dt\int_{\pi/N}^{2\pi/N}\sin Nx\,dx\sum_{k=1}^{(N-1)/2}\frac{f(x+t+(2k-1)\pi/N)-f(x+t+2k\pi/N)}{(2k-1)\pi/N}$$
$$|J_1|\leqq\int_0^{2\pi}|g(t)|dt\cdot N\int_{\pi/N}^{2\pi/N}\left|\sum_{k=1}^{(N-1)/2}\frac{f(x+t+(2k-1)\pi/N)-f(x+t+2k\pi/N)}{(2k-1)\pi/N}\right|dx$$
$$=o(1)$$

since $g\in L(0,2\pi)$ and the inner integral is uniformly bounded and tends to zero almost everywhere as $N\to\infty$ by the assumption of the theorem. It remains now to prove that $J_2-J_3=o(1)$ as $N\to\infty$.

$$J_2=\sum_{k=1}^{(N-1)/2}\int_{\pi/N}^{2\pi/N}\sin Nx\,dx$$
$$\frac{x+\pi/N}{((2k-1)\pi/N)(x+2k\pi/N)}\int_0^{2\pi}g(t)[f(x+2k\pi/N+t)-f(t)]dt,$$
$$|J_2|\leqq A\max_{\pi/N\leqq x\leqq 2\pi/N}\sum_{k=1}^{(N-1)/2}\frac{1}{k^2}\int_0^{2\pi}|g(t)|\,|f(x+2k\pi/N+t)-f(t)|\,dt$$
$$\leqq A\max_{\pi/N\leqq x\leqq 2\pi/N}\left(\sum_{k=1}^{N_0}+\sum_{k=N_0+1}^{(N-1)/2}\right)=J_{21}+J_{22},$$

where N_0 is a large but fixed integer.

For any $\varepsilon>0$, we can write $g(t)=g_1(t)+g_2(t)$, where $|g_1(t)|\leqq M_1$ and $\int_0^{2\pi}|g_2(t)|\,dt<\varepsilon$. Then

$$J_{21}\leqq A \max_{\pi/N\leqq x\leqq 2\pi/N} \sum_{k=1}^{N_0}\frac{M_1}{k}\int_0^{2\pi}|f(x+2k\pi/N+t)-f(t)|\,dt$$
$$+A\sum_{k=1}^{N_0}\frac{M_2}{k^2}\int_0^{2\pi}|g_2(t)|\,dt\leqq A\varepsilon,$$

where $|f(x)|\leqq M_2$,

$$J_{22}\leqq A\,M_2\sum_{k=N_0+1}^{\infty}\frac{1}{k^2}\leqq\frac{A}{N_0}$$

which is less than ε for large N_0. Thus we have proved that J_2 tends to zero as $N\to\infty$. Similarly $J_3=o(1)$.

3. **Theorem 2.** *If $g(t)$ is bounded measurable and $f(t)$ is an integrable function such that*

$$(3)\qquad \int_0^{2\pi}\left|\sum_{k=1}^{[N/2]}\frac{f(t\pm 2k\pi/N)-f(t\pm(2k-1)\pi/N)}{k}\right|dt$$

tends to zero as $N\to\infty$, then the Parseval relation (1) holds, the right side series being convergent.

Proof. As in the proof of Theorem 1, we divide the integral form of $\pi^2\Big(S_N-\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}f(x)g(x)dx\Big)$ into I, J_1, J_2, J_3. We can easily prove that $I=o(1)$ and $J_2+J_3=o(1)$. Concerning J_1 we get

$$\sum_{k=1}^{(N-1)/2}\int_{\pi/N}^{2\pi/N}\sin Nx\,dx\Big[\frac{1}{x+(2k-1)\pi/N}$$
$$\int_0^{2\pi}g(t)\Big[f\Big(x+t+\frac{(2k-1)\pi}{N}\Big)-f\Big(x+t+\frac{2k\pi}{N}\Big)\Big]dt\Big]$$
$$=\int_0^{2\pi}g(t)dt\Big[\int_{\pi/N}^{2\pi/N}\sin Nx\sum_{k=1}^{(N-1)/2}\frac{f(x+t+(2k-1)\pi/N)-f(x+t+2k\pi/N)}{x+(2k-1)\pi/N}dx\Big]$$
$$|J_1|\leqq\int_0^{2\pi}dt\left|\int_{\pi/N}^{2\pi/N}\sin Nx\sum_{k=1}^{(N-1)/2}\frac{f(x+t+(2k-1)\pi/N)-f(x+t+2k\pi/N)}{(2k-1)\pi/N}dx\right|$$
$$=\int_0^{2\pi}dt\left|\int_{t+\pi/N}^{t+2\pi/N}\sin N(x-t)\sum_{k=1}^{(N-1)/2}\frac{f(x+(2k-1)\pi/N)-f(x+2k\pi/N)}{(2k-1)\pi/N}dx\right|$$
$$\leqq\int_0^{2\pi}dt\int_{t+\pi/N}^{t+2\pi/N}\left|\sum_{k=1}^{(N-1)/2}\frac{f(x+(2k-1)\pi/N)-f(x+2k\pi/N)}{(2k-1)\pi/N}\right|dx$$
$$\leqq\int_0^{2\pi}\left|\sum_{k=1}^{(N-1)/2}\frac{f(x+(2k-1)\pi/N)-f(x+2k\pi/N)}{(2k-1)\pi}\right|dx$$

which tends to zero as $N\to\infty$ by the assumption. Thus the theorem is proved.

Especially, if

$$\int_0^{2\pi}|f(x)-f(x+\pi/N)|\,dx=o(1/\log N),$$

then the assumption (3) is satisfied. This is also a special case of Theorem II.

4. We shall state theorems which can be proved by the above method. The proof may be omitted.

1° If $g(t)$ is of bounded variation and $f(x)$ is a continuous function such that (2) (or (3)) satisfies the assumption in Theorem 1 (or Theorem 2), then the Parseval relation

$$\int_0^{2\pi} f(t)dg(t)=\frac{a_0a_0'}{4}+\sum_{n=1}^{\infty}(a_na_n'+b_nb_n')$$

holds where a_n, b_n are Fourier coefficients of $f(t)$ and a_n', b_n' are Fourier-Stieltjes coefficients of $g(t)$.

2° If $g(t)$ is integrable and $f(t)$ is a bounded measurable function such that

$$N\int_{x+\pi/N}^{x+2\pi/N}\sum_{k=1}^{[(N-1)/2]}\frac{|f(t\pm 2k\pi/N)-f(t\pm(2k-1)\pi/N)|}{k^{1-\alpha}}dt \qquad (\alpha>0)$$

is bounded for all x and tends to zero as $N\to\infty$, then the Parseval relation (1) holds, where the right side series is $(C, -\alpha)$ summable (cf. [4, 5]).

3° If $g(t)$ is bounded measurable and $f(t)$ is an integrable function belonging to the class lip $(1, \alpha)$, then the Parseval relation (1) holds where the right side series is summable $(C, -\alpha)$ (cf. [4, 5]).

References

[1] S. Izumi and M. Satô: Some trigonometrical series. XVIII, Proc. Japan Acad., **32** (1956).

[2] M. Satô: Uniform convergence of Fourier series. V, Proc. Japan Acad., **31** (1955).

[3] R. Salem: La convergence uniforme des séries de Fourier, Comptes Rendus, **207** (1938).

[4] S. Izumi and T. Kawata: Notes on Fourier series IX, Tôhoku Mathematical Journal, **46** (1939).

[5] S. Izumi: Some trigonometrical series IX, Tôhoku Mathematical Journal, **6** (1953).