100. Contributions to the Theory of Semi-groups. IV

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Following G. Thierrin [7], a semi-group S is called *strongly reversible*, if, for any two elements a, b of S, there are three positive integers r, s and t such that

$$(ab)^r = a^s b^t = b^t a^s.$$

Such a notion is a generalisation of a commutative semi-group.

In this paper, we are mainly concerned with generalisations of the results by S. Schwarz [4–6].

Let $\mathfrak{A}$ be a two-sided ideal of S. We denote by $\overline{\mathfrak{A}}$ the set of element a such that $a^s \in \mathfrak{A}$ for some positive integer s. $\overline{\mathfrak{A}}$ is called the *closure* of $\mathfrak{A}$.

Theorem 1. If a semi-group S is strongly reversible, the closure $\overline{\mathfrak{A}}$ of any two-sided ideal $\mathfrak{A}$ is a two-sided ideal.

Proof. Let a be an element of $\overline{\mathfrak{A}}$ and let x be an element of S. Then there is a positive integer k such that $a^k \in \mathfrak{A}$, and there are three integers r, s and t such that

$$(ax)^r = a^s x^t = x^t a^s.$$

Hence, we have

$$(ax)^{rk} = (a^s x^t)^k = a^{sk} x^{tk} \in \mathfrak{A} x^{tk} \subseteq \mathfrak{A}.$$

Thus $ax \in \overline{\mathfrak{A}}$. Similarly $xa \in \overline{\mathfrak{A}}$. Therefore, $\overline{\mathfrak{A}}$ is a two-sided ideal.

A semi-group S is called a *periodic semi-group*, if, for every element a of S, the semi-group (a) generated by a contains a finite number of different elements.

Such a semi-group has been extensively studied by S. Schwarz.

Theorem 2. Let $\mathfrak{A}$ be a two-sided ideal of a strongly reversible periodic semi-group S, and let $\{e_\alpha\}$ be the set of all idempotents of $\mathfrak{A}$, then

$$\overline{\mathfrak{A}} = \bigcup_\alpha K^{(\alpha)},$$

where $K^{(\alpha)}$ is the largest subsemi-group of S containing only one idempotent e_α.

For the detail of the semi-group $K^{(\alpha)}$, see K. Iséki [3].

Proof. Let $a \in K^{(\alpha)}$, then $a^s = e_\alpha$ for some s. Hence $a \in \overline{\mathfrak{A}}$ and we have $\bigcup_\alpha K^{(\alpha)} \subseteq \overline{\mathfrak{A}}$. Conversely, let $a \in \overline{\mathfrak{A}}$, then $a^s \in \mathfrak{A}$ for some s. Hence there is an integer t such that $(a^s)^t = e_\alpha \in K^{(\alpha)}$. This shows $a \in K^{(\alpha)}$.

Therefore $$\bar{\mathfrak{A}} \subseteq \bigcup_{\alpha} K^{(\alpha)}.$$

Definition 1. If $\bar{\mathfrak{A}}=\mathfrak{A}$, $\mathfrak{A}$ is called a *closed ideal*. A two-sided ideal $\mathfrak{P}$ is called *prime*, if $ab \in \mathfrak{P}$ implies $a \in \mathfrak{P}$ or $b \in \mathfrak{P}$, equivalently, $S-\mathfrak{P}$ is a semi-group:

Theorem 3. The intersection of any number of prime ideals $\mathfrak{P}_\alpha$ is closed.

Proof. Let $\mathfrak{A}=\bigcap_{\alpha} \mathfrak{P}_\alpha$, then it is clear that $\mathfrak{A} \subseteq \bar{\mathfrak{A}}$. For a of $\bar{\mathfrak{A}}$, there is a positive integer s such that

$$a^s \in \mathfrak{A} = \bigcap_{\alpha} \mathfrak{P}_\alpha.$$

Hence $a^s \in \mathfrak{P}_\alpha$ for every α. Since every $\mathfrak{P}_\alpha$ is prime, $a \in \mathfrak{P}_\alpha$. Therefore, $a \in \bigcap_{\alpha} \mathfrak{P}_\alpha = \mathfrak{A}$. So we have $\mathfrak{A} \subseteq \mathfrak{A}$. Hence $\mathfrak{A}$ is closed.

We shall prove the following theorem which is a generalisation of S. Schwarz theorem [6].

Theorem 4. Any closed ideal of a commutative periodic semi-group is the intersection of some prime ideals.

Proof. By Theorem 2, we have

$$\mathfrak{A}=\bar{\mathfrak{A}}= \bigcup_{\alpha \in \Lambda_1} K^{(\alpha)},$$

where $e_\alpha (\alpha \in \Lambda_1)$ is the set of all idempotents in $\mathfrak{A}$. By Theorem 2 of my paper [2], there is a disjoint decomposition:

$$S = \bigcup_{\alpha \in \Lambda_1} K^{(\alpha)} \smile \bigcup_{\beta \in \Lambda_2} K^{(\beta)}.$$

Then we can find by Zorn's lemma the family of ideals $\mathfrak{P}_\beta (\beta \in \Lambda_2)$ satisfying the following conditions:

(1) $\mathfrak{A} \subset \mathfrak{P}_\beta$,

(2) $\mathfrak{P}_\beta \cap K^{(\beta)} = 0$,

(3) $\mathfrak{P}_\beta$ is maximal with respect to the condition (2).

From the conditions (1) and (2), we have easily

$$\mathfrak{A} = \bigcap_{\beta \in \Lambda_2} \mathfrak{P}_\beta.$$

Therefore, $\mathfrak{A}$ is the intersection of $\mathfrak{P}_\beta (\beta \in \Lambda_2)$. (For such a consideration, see K. Iséki [2] or K. E. Aubert [1].)

Now we shall show that each $\mathfrak{P}_\beta$ is a prime ideal. The idea of the proof is due to S. Schwarz [6]. To prove it, let $a, b \in S-\mathfrak{P}_\beta$, then, by the condition (3), the ideal $\{\mathfrak{P}_\beta, a, aS\}$ meets $K^{(\beta)}$. Similarly $\{\mathfrak{P}_\beta, b, bS\} \frown K^{(\beta)} \neq 0$. From $K^{(\beta)} \frown \mathfrak{P}_\beta = 0$, we have

$$\{a, Sa\} \frown K^{(\beta)} \neq 0 \neq \{b, bS\} \frown K^{(\beta)}.$$

If $a \in K^{(\beta)}$ and $b \in K^{(\beta)}$, then we have $ab \in K^{(\beta)} \subset S-\mathfrak{P}_\beta$.

If $a \in K^{(\beta)}$ and $bx \in K^{(\beta)}$, then $abx \in K^{(\beta)}$ and since $\mathfrak{P}_\beta$ is an ideal, $ab \in \mathfrak{P}_\beta$. Hence $ab \in S-\mathfrak{P}_\beta$.

If $ax \in K^{(\beta)}$ and $by \in K^{(\beta)}$, by a similar argument, we have $ab \in S-\mathfrak{P}_\beta$.

Therefore, each $\mathfrak{P}_\beta$ is a prime ideal.

Corollary 1. In any commutative periodic semi-group, an ideal is closed, if and only if, it is the intersection of some prime ideals.

Let S be a strongly reversible periodic semi-group, then by the decomposition theorem (see K. Iséki [2]), we have

$$S=\bigcup_{\alpha} K^{(\alpha)}$$

where α runs over all idempotents e_α of S, and $K^{(\alpha)}\frown K^{(\beta)}=0$ for $\alpha\neq\beta$. Each $G^{(\alpha)}=K^{(\alpha)}e_\alpha=e_\alpha K^{(\alpha)}$ is a group. An element of $G^{(\alpha)}$ is called *regular*. It is easily seen that the set of all regular elements of a commutative periodic semi-group is a commutative semi-group.

The set E of all idempotents of a strongly reversible periodic semi-group is a commutative semi-group. Let E_1 be an ideal of E, then $J=\bigcup_{e_\beta\in E} K^{(\beta)}$ is a two-sided ideal of S. To prove it, let $a\in J$ and $x\in S$. Suppose that $a\in K^{(\alpha)}\subset J$, then there are two natural numbers ρ, τ such that

$$a^\rho=e_\alpha,\ x^\tau=e_\omega.$$

On the other hand, we can find three positive integers r, s and t such that

$$(ax)^r=a^s x^t=x^t a^s.$$

Hence

$$(ax)^{r\rho\tau}=(a^s)^{\rho\tau}(x^t)^{\rho\tau}=e_\alpha e_\omega\in E_1\cdot E\subseteq E_1.$$

Therefore, ax is contained in some $K^{(\gamma)}$ of J. Similarly xa is contained in some $K^{(\gamma')}$ of J. Hence J is an ideal of S.

Now let E_1 be a prime ideal of E. Then we shall show that J is a prime ideal of S.

Let $a, b\in S-J$, then there are ρ, τ such that

$$a^\rho=e_\xi,\qquad b^\tau=e_\eta$$

and $e_\xi e_\eta\in E-E_1$. Since S is strongly reversible, there are three natural numbers r, s and t such that

$$(ab)^r=a^s b^t=b^t a^s.$$

Hence

$$(ab)^{r\rho\tau}=e_\xi e_\eta\in E-E_1.$$

Therefore $ab\in S-J$.

If J is a two-sided ideal of S, the set E_1 of all idempotents in J is an ideal of E. If J is a prime ideal of S, then we can prove that E_1 is a prime ideal of E.

On the first part, from $E_1\subseteq J$, we have

$$E_1\cdot E\subseteq J\cdot S\subseteq J,$$

and since each element of E_1E is idempotent of J, $E_1E\subseteq E_1$. Now, let J be a prime ideal of S, and let $e_\xi e_\eta\in E-E_1$, then $e_\xi e_\eta\in S-J$. Hence, we have $e_\xi e_\eta\in S-J$. Since $e_\xi e_\eta$ is an idempotent of S, $e_\xi e_\eta\in E-E_1$, which completes the proof.

If a prime ideal J_1 of S is distinct from a prime ideal J_2, then $J_1-J_2\neq 0$ or $J_2-J_1\neq 0$. Suppose that $J_1-J_2\neq 0$, then there is an element a such that $a\in J_1$ and $a\bar{\in} J_2$. If $a^\rho=e_\alpha$, $e_\alpha\bar{\in} J_2$ and $e_\alpha\bar{\in} J_2$ since J_2 is a prime ideal. Hence $E_1\neq E_2$.

Therefore, we have the following

Theorem 5. In any strongly reversible periodic semi-group,

1) *the set E of all idempotents of S is commutative semi-group;*

2) *an ideal of E corresponds to a two-sided ideal of S and its converse;*

3) *there is a $1-1$ correspondence between the collection of prime ideals in E and the set of prime ideals in S.*

Let E be the set of all idempotents of a semi-group S. If $e_\alpha e_\beta=e_\alpha$ for $e_\alpha, e_\beta\in E$, we write $e_\alpha\leq e_\beta$. This order defines a quasi-order on E. If E is commutative, E is a partially ordered set on "$\leq$" (see S. Schwarz [4]).

In any commutative periodic semi-group S, if $e_\alpha\geq e_\beta$, $a\to ae_\beta$ $(a\in K^{(\alpha)})$ is a homomorphic mapping from $K^{(\alpha)}$ to $G^{(\beta)}$ (see S. Schwarz [4]).

Theorem 6. In a strongly reversible periodic semi-group S, if $\mathfrak{P}$ is a prime ideal of S, and $e'\leq e$, $e\in\mathfrak{P}$, $e, e'\in E$, then $e'\in\mathfrak{P}$.

Proof. $e'\leq e$ implies $e'=e'e$, and by $e\in\mathfrak{P}$, we have $e'\in\mathfrak{P}$.

By a *character* of a semi-group S, we mean a complex valued function $\chi(x)$ satisfying $\chi(a)\chi(b)=\chi(ab)$ for every a, b of s.

The following propositions are clear.

Proposition A. Let $\chi(x)$ be a character of S, then the set $\{x\mid\chi(x)=0\}$ is a prime ideal of S.

Proposition B. Let $\mathfrak{P}$ be a prime ideal of S, then

$$\varepsilon_{\mathfrak{P}}(x)=\begin{cases}0 & x\in\mathfrak{P}\\ 1 & x\in S-\mathfrak{P}\end{cases}$$

is a character of S.

Proposition C. Let e be an idempotent of S, then $\chi(e)=0$ or $\chi(e)=1$.

Proposition D. The set $\hat{S}$ of all characters of S is a commutative semi-group with 0 and a unit.

For χ, ψ of $\hat{S}$, the product $\chi\psi$ is defined $\chi\psi(a)=\chi(a)\psi(a)$ for all $a\in S$.

Proposition E. The character $\varepsilon_{\mathfrak{P}}(x)$ for a prime ideal $\mathfrak{P}$ is an idempotent of $\hat{S}$.

Proposition F. Let $\mathfrak{P}$ be a prime ideal of S. The set of all characters which vanish just on $\mathfrak{P}$ forms a group $\hat{G}_{\mathfrak{P}}$ with $\varepsilon_{\mathfrak{P}}$ as the unit element (see S. Schwarz [4], p. 226).

Proposition G. $\hat{S}$ can be written as a sum of disjoint groups $\hat{G}_{\mathfrak{P}}$

for all prime ideals and the set $G_{\emptyset}$ of all non-vanishing characters.

Proposition H. Any ideal of $\hat{S}$ is closed.

Proof. By Proposition G, $\hat{S}$ is a set sum of disjoint groups $G_{\mathfrak{P}}$. Let $\mathfrak{A}$ be an ideal of $\hat{S}$, and let χ be an element of $\mathfrak{A}$, then there is a group $G_{\mathfrak{P}}$ containing χ and hence $G_{\mathfrak{P}}\chi\subset\mathfrak{A}$. Since $G_{\mathfrak{P}}$ is a group, $G_{\mathfrak{P}}\chi=G_{\mathfrak{P}}$. Therefore $G_{\mathfrak{P}}\subseteq\mathfrak{A}$. Hence $\mathfrak{A}$ is written as a set sum of some $G_{\mathfrak{P}}$. Let χ be an element of $\mathfrak{A}$, then $\chi^n\in\mathfrak{A}$ for some n. Therefore there is a group $G_{\mathfrak{P}}$ containing χ^n. If $\chi\in G_{\mathfrak{P}'}$ and $\mathfrak{P}\neq\mathfrak{P}'$, then $\chi^n\in G_{\mathfrak{P}'}$. Hence $\chi^n\,\bar{\in}\, G_{\mathfrak{P}}$. This shows that $\chi^n\in G_{\mathfrak{P}}$ implies $\chi\in G_{\mathfrak{P}}$. Hence $\chi\in\mathfrak{A}$. Therefore $\mathfrak{A}$ is closed.

In his paper [5], S. Schwarz has studied the structures of a character of a commutative finite semi-group. We shall generalize his results to some general classes containing commutative finite semi-groups.

Let S be a commutative periodic semi-group with a least idempotent. Following S. Schwarz [5], we shall define a conjugate class. Let $G^{(\alpha)}$ be the maximal subgroup of $K^{(\alpha)}$ for an idempotent e_α. For a of $G^{(\alpha)}$, the set T_a of all elements x of $K^{(\alpha)}$ such that $xe_\alpha=a$ is called a *conjugate class* of S. The semi-group S is a set sum of disjoint conjugate classes.

Let a, b be different elements of S. Every $K^{(\alpha)}$ is the sum of some conjugate classes.

Suppose that $a, b\in K^{(\alpha)}$, then $ae_\alpha\neq be_\alpha$. We shall divide the set E of all idempotents with disjoint classes as follows: let $E_1=\{e\mid ee_\alpha=e_\alpha\}$. $E_2=\{e\mid ee_\alpha\neq e_\alpha\}$, then the sets E_1, E_2 are disjoint and $E=E_1\cup E_2$. We shall show that E_2 is a prime ideal of E. If e, e' are in E_1, then $ee_\alpha=e_\alpha$ and $e'e_\alpha=e_\alpha$, hence $ee'e_\alpha=ee_\alpha=e_\alpha$. Therefore $ee'\in E_1$ and hence E_1 is a semi-group. To prove that E_2 is an ideal, let $e\in E_2$, $e'\in E$, then $ee_\alpha\neq e_\alpha$. To prove $ee'e_\alpha\neq e_\alpha$, suppose $ee'e_\alpha=e_\alpha$, then we have $ee_\alpha=eee'e_\alpha=ee'e_\alpha=e_\alpha$, which is a contradiction to $ee_\alpha\neq e_\alpha$. Hence E_2 is an ideal of E. Therefore E_2 is a prime ideal of E. Hence $\mathfrak{P}=\bigcup_{e_\beta\in E_2}K^{(\beta)}$ is a prime ideal of S, by Theorem 5. From $K^{(\alpha)}\ni a, b$, we have $ae_\alpha, be_\alpha\in G^{(\alpha)}$. Since $G^{(\alpha)}$ is a discrete commutative group, there is a character $\chi(x)$ of $G^{(\alpha)}$ such that $\chi(ae_\alpha)\neq\chi(be_\alpha)$. Clearly $\chi(e_\alpha)\neq 0$. Let

$$\psi(x)=\begin{cases}0 & x\in\mathfrak{P}\\ \chi(xe_\alpha) & x\in K^{(\alpha)},\end{cases}$$

since e_α is a least idempotent of E_1, and S is commutative, $x\to xe_\alpha$ ($x\in K^{(\beta)}$, and $e_\beta\in E_1$) is a homomorphic mapping from $K^{(\beta)}$ to $G^{(\alpha)}$, $\psi(x)$ is extended on $S-\mathfrak{P}$ by $\psi(x)=\psi(xe_\alpha)$ ($x\in K^{(\beta)}$ and $e_\beta\in E_1$). Therefore $\psi(x)$ is a character of S and $\chi(a)\neq\chi(b)$.

Next, we shall suppose $a \in K^{(\alpha)}$, $b \in K^{(\beta)}$ and $e_\alpha \neq e_\beta$. If $e_\alpha < e_\beta$, then $e_\alpha = e_\alpha e_\beta$, and let $E_1 = \{e \mid ee_\beta = e_\beta\}$, and $E_2 = E - E_1$. If $e_\alpha \in E_1$, then $e_\alpha e_\beta = e_\beta$ and hence $e_\alpha = e_\beta$. Therefore $E_2 \ni e_\alpha$. By the similar method stated above, we can show that $\mathfrak{P} = \bigcup_{e_\gamma \in E_1} K^{(\gamma)}$ is a prime ideal, and hence $e_\alpha \in S - \mathfrak{P}$, $e_\beta \in \mathfrak{P}_\beta$. Let

$$\chi(x) = \begin{cases} 1 & \text{for } x \in \mathfrak{P} \\ 0 & \text{for } x \in S - \mathfrak{P}, \end{cases}$$

then $\chi(x)$ is a character of S and $\chi(a) = 0$, $\chi(b) = 1$.

If $e_\alpha \not< e_\beta$, $e_\beta \not< e_\alpha$, let E_1 be the set of all idempotents greater than e_α, i.e. $E_1 = \{e \mid ee_\alpha = e_\alpha, e \in E\}$, then e_β is not contained in E_1. Therefore, by the above method, we can construct a character $\chi(x)$ of S such that $\chi(a) = 1$ and $\chi(b) = 0$. Therefore we have proved the following

Theorem 7. In any commutative periodic semi-group, for any two elements a, b from distinct conjugate classes, there is a character $\chi(x)$ of S such that $\chi(a) \neq \chi(b)$.

Theorem 7 is a generalisation of the result by S. Schwarz [5].

References

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