99. *Notes on Topological Spaces. V. On Structure Spaces of Semiring*

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The structure space of maximal ideals in a normed ring has been discussed by I. Gelfand and G. Silov [2]. Recently, the structure space of maximal ideals in a semiring has been considered by W. Slowikowski and W. Zawadowski [4] and the theory has generalized by the present author and Y. Miyanaga [3]. E. A. Behrens [1] has considered the relation of structure spaces formed by three special classes of ideals of a naring.

In this Note, we shall consider the *structure space* $\mathfrak{P}$ of all prime ideals of a semiring A and the relation of $\mathfrak{P}$ and the structure space $\mathfrak{M}$ of all maximal ideals of A. Throughout the paper, we shall treat a commutative semiring A with a zero 0 and a unit 1. (For detail of the definition, see K. Iséki and Y. Miyanaga [3].)

An ideal P of A is *prime* if and only if $ab \in P$ implies $a \in P$ or $b \in P$. Since A has a unit 1, any maximal ideal is prime, therefore $\mathfrak{P} \supseteq \mathfrak{M}$.

To introduce a topology γ on $\mathfrak{P}$, we shall take $\gamma_x = \{P \mid x \,\bar{\in}\, P, P \in \mathfrak{P}\}$ for every x of A as an open base of $\mathfrak{P}$. Then we have the following

Theorem 1. Let $\mathfrak{A}$ be a subset of $\mathfrak{P}$, then

$$\bar{\mathfrak{A}} = \{P' \mid \bigcap_{P \in \mathfrak{A}} P \subset P' \ \text{ and } \ P' \in \mathfrak{P}\},$$

where $\bar{\mathfrak{A}}$ is the closure of $\mathfrak{A}$ by the topology γ.

Proof. To prove that the $\bar{\mathfrak{A}}$ contains $\{P' \mid \bigcap_{P \in \mathfrak{A}} P \subset P', P' \in \mathfrak{P}\}$, let $P' \in \{P' \mid \bigcap_{P \in \mathfrak{A}} P \subset P', P' \in \mathfrak{A}\}$, and let γ_x be a neighbourhood of P', then $x \,\bar{\in}\, P'$, and we have $x \,\bar{\in} \bigcap_{P \in \mathfrak{A}} P$. Therefore, there is a prime ideal $P \in \mathfrak{A}$ such that P does not contain x and $\gamma_x \ni P$. This shows $P \in \bar{\mathfrak{A}}$.

If a prime ideal P' is not in $\{P' \mid \bigcap_{P \in \mathfrak{A}} P \subset P', P' \in \mathfrak{A}\}$, then $\bigcap_{P \in \mathfrak{A}} P - P'$ is not empty. Hence, for $x \in \bigcap_{P \in \mathfrak{A}} P - P'$, we have $x \in P, P \in \mathfrak{A}$ and $x \,\bar{\in}\, P'$. This shows $\gamma_x \,\bar{\ni}\, P$, $P \in \mathfrak{A}$ and $\gamma_x \,\bar{\ni}\, P'$. Therefore $\gamma_x \frown \mathfrak{A} = 0$ and hence $P' \,\bar{\in}\, \bar{\mathfrak{A}}$. The proof is complete.

A similar argument for $\mathfrak{M}$ relative to Γ-topology implies the following

Proposition. Let $\mathfrak{A}$ be a subset of $\mathfrak{M}$, then

$$\overline{\mathfrak{A}}=\{M' \mid \bigcap_{M\in\mathfrak{A}} M\subset M', \text{ and } M'\in\mathfrak{M}\}$$

where $\overline{\mathfrak{A}}$ *is the closure of* $\mathfrak{A}$ *by the topology* Γ.

The proposition has been also proved by W. Slowikowski and W. Zawadowski [4].

The closure operation $\mathfrak{A}\to\overline{\mathfrak{A}}$ *of* $\mathfrak{P}$ *satisfies the following relations*:

(1) $\mathfrak{A}\subseteq\overline{\mathfrak{A}}$

(2) $\overline{\mathfrak{A}}=\overline{\overline{\mathfrak{A}}}$

(3) $\overline{\mathfrak{A}}\smile\overline{\mathfrak{B}}=\overline{\mathfrak{A}\smile\mathfrak{B}}$.

We shall prove only the last relation (3). By Theorem 1, $\mathfrak{A}\subset\mathfrak{B}$ implies $\overline{\mathfrak{A}}\subset\overline{\mathfrak{B}}$, and hence $\overline{\mathfrak{A}}\smile\overline{\mathfrak{B}}\subset\overline{\mathfrak{A}\smile\mathfrak{B}}$. Let $P\,\overline{\in}\,\overline{\mathfrak{A}}\smile\overline{\mathfrak{B}}$, then $P\,\overline{\in}\,\overline{\mathfrak{A}}$ and $P\,\overline{\in}\,\overline{\mathfrak{B}}$. Hence $P\not\supset\bigcap_{P'\in\mathfrak{A}}P'=P_{\mathfrak{A}}$ and $P\not\supset\bigcap_{P'\in\mathfrak{B}}P'=P_{\mathfrak{B}}$. The sets $\mathfrak{P}_{\mathfrak{A}}$ and $\mathfrak{P}_{\mathfrak{B}}$ are ideals. If $P_{\mathfrak{A}}P_{\mathfrak{B}}\subset P$, for any elements a, b such that $a\in P_{\mathfrak{A}}-P$, $b\in P_{\mathfrak{B}}-P$, we have $ab\in P$ and since P is a prime ideal, $a\in P$ or $b\in P$, which is a contradiction. Therefore $P\not\supset P_{\mathfrak{A}}\cdot P_{\mathfrak{B}}\supseteq P_{\mathfrak{A}}\frown P_{\mathfrak{B}}=P_{\mathfrak{A}\cup\mathfrak{B}}$. Hence $P\,\overline{\in}\,\overline{\mathfrak{A}\smile\mathfrak{B}}$.

Theorem 2. The topological space $\mathfrak{P}$ *is a* T_0*-space.*

Proof. It is sufficient to prove that $(\overline{P_1})=(\overline{P_2})$ implies $P_1=P_2$. By $P_2\in(\overline{P_1})$, then $P_2\supset P_1$. Similarly $P_1\supset P_2$ and we have $P_1=P_2$. We can easily prove the following

Theorem 3. Every prime ideal is maximal, if and only if the topological space $\mathfrak{P}$ *is a* T_1*-space.*

In our previous Note [3], we have proved the following

Theorem 4. The topological space $\mathfrak{M}_\Gamma$ *is a compact* T_1*-space.*

Now we shall prove that $\mathfrak{P}$ is a compact space. Let $\mathfrak{A}_\lambda$ be a family of closed sets such that $\bigcap_\lambda\mathfrak{A}_\lambda=0$, then we have $\Sigma P_{\mathfrak{A}_\lambda}=A$, where $P_{\mathfrak{A}_\lambda}=\bigcap_{P\in\mathfrak{A}_\lambda}P$. Suppose that $\Sigma P_{\mathfrak{A}_\lambda}\neq A$, then there is a maximal ideal M containing $\Sigma P_{\mathfrak{A}_\lambda}$. Therefore $\Sigma P_{\mathfrak{A}_\lambda}\subset M$ for every λ. Hence $\mathfrak{A}_\lambda\ni M$ for every λ, and we have $\bigcap_\lambda\mathfrak{A}_\lambda\ni M$, which is a contradiction. By $\Sigma P_{\mathfrak{A}_\lambda}=A$, the unit 1 may be expressed by a form $a_1+a_2+\cdots+a_n=1$, $a_i\in P_{\mathfrak{A}_{\lambda_i}}$ $(i=1,2,\cdots,n)$. Hence $A=\sum_{i=1}^n P_{\mathfrak{A}_{\lambda_i}}$. If $\bigcap_{i=1}^n\mathfrak{A}_{\lambda_i}\neq 0$, then for a prime ideal P of $\bigcap_{i=1}^n\mathfrak{A}_{\lambda_i}$, we have $P\supset P_{\mathfrak{A}_{\lambda_i}}$ $(i=1,2,\cdots,n)$ and hence $P\supset\sum_{i=1}^n P_{\mathfrak{A}_{\lambda_i}}$. Therefore we have $\bigcap_{i=1}^n\mathfrak{A}_{\lambda_i}=0$. Hence we have the following

Theorem 5. The topological space $\mathfrak{P}$ *is a compact* T_1*-space.*

Definition 1. By the $\mathfrak{P}$*-radical* $r(\mathfrak{P})$ of A, we mean the intersection of all prime ideals of A, i.e. $\bigcap_{P\in\mathfrak{P}}P$. By the $\mathfrak{M}$*-radical* $r(\mathfrak{M})$

of A, we mean the intersection of all maximal ideals of A, i.e. $\bigcap_{M \in \mathfrak{M}} M$.

From $\mathfrak{M} \subseteq \mathfrak{P}$, we have $r(\mathfrak{P}) \subseteq r(\mathfrak{M})$. Next we shall give a topological condition to be $r(\mathfrak{P}) = r(\mathfrak{M})$.

Theorem 6. The subset $\mathfrak{M}$ of $\mathfrak{P}$ is dense in $\mathfrak{P}$, if and only if, $r(\mathfrak{P}) = r(\mathfrak{M})$.

Proof. Let $\overline{\mathfrak{M}} = \mathfrak{P}$ for the topology γ, then we have

$$\{P \mid \bigcap_{M \in \mathfrak{M}} M \subset P\} = \mathfrak{P}.$$

Hence

$$r(\mathfrak{M}) = \bigcap_{M \in \mathfrak{M}} M \subseteq \bigcap_{P \in \mathfrak{P}} P = r(\mathfrak{P}).$$

By the remark above, $r(\mathfrak{M}) \supseteq r(\mathfrak{P})$. Therefore $r(\mathfrak{M}) = r(\mathfrak{P})$.

Conversely, if $P \in \mathfrak{P} - \overline{\mathfrak{M}}$, then $P \in \mathfrak{P}$ and $P \in \overline{\mathfrak{M}}$. Therefore, there is a neighbourhood γ_x of P such that $\gamma_x \frown \mathfrak{M} = 0$. Hence $r(\mathfrak{P}) = \bigcap_{P \in \mathfrak{P}} P$ is a proper subset of $\bigcap_{M \in \mathfrak{M}} M$. Therefore $r(\mathfrak{P}) \neq r(\mathfrak{M})$, which completes the proof.

Definition 2. If $r(\mathfrak{M}) = 0$, A is said to be *semi-simple.*

From Theorem 6, we have the following

Theorem 7. If A is semi-simple, $\mathfrak{M}$ is dense in $\mathfrak{P}$.

Definition 3. If $r(\mathfrak{P})$ is the zero-ideal (0), A is said to be *p-semi-simple.*

Then we shall prove the following

Theorem 8. Let A be a semi-simple semiring. A is the direct sum of two semi-simple semirings, if and only if $\mathfrak{M}$ is separated.

Theorem 9. Let A be a p-semi-simple semiring. A is the direct sum of two p-semi-simple semirings, if and only if $\mathfrak{P}$ is separated.

The Proof of Theorem 9. Let A be the direct sum of p-semi-simple semirings A_1 and A_2, i.e. $A = A_1 \oplus A_2$. Let $\mathfrak{A}_i = \{P \mid A_i \subset P, P \in \mathfrak{P}\}$, then $\mathfrak{A}_1, \mathfrak{A}_2$ are closed in $\mathfrak{P}$. Since A_i is p-semi-simple, $A_i = \bigcap_{P \in \mathfrak{A}_i} P$ $(= P_{\mathfrak{A}_i})$ and hence

$$A = A_1 \oplus A_2 = P_{\mathfrak{A}_1} \oplus P_{\mathfrak{A}_2}.$$

Therefore $\mathfrak{A}_1 \frown \mathfrak{A}_2 = 0$. To prove that $\mathfrak{P} = \mathfrak{A}_1 \smile \mathfrak{A}_2$, if there is a prime ideal P of $\mathfrak{P}$ such that $P \overline{\in} \mathfrak{A}_1$ and $P \overline{\in} \mathfrak{A}_2$, then P does not contain A_1 and A_2. On the other hand, $A_1 \cdot A_2 \subseteq A_1 \frown A_2 = (0) \subseteq P$, hence $A_1 \subset P$ or $A_2 \subset P$, which is a contradiction.

Conversely, let $\mathfrak{P} = \mathfrak{A}_1 \smile \mathfrak{A}_2$, $\mathfrak{A}_1 \frown \mathfrak{A}_2 = \phi$ and let $\mathfrak{A}_1, \mathfrak{A}_2$ be non-empty and closed in $\mathfrak{P}$, then $A_i = \bigcap_{P \in \mathfrak{A}_i} P$ $(i=1, 2)$ are ideals, and since A is p-semi-simple, $A_1 \frown A_2 = (0)$ and we have $A_1 \oplus A_2 = A$. From $A_i = \bigcap_{P \in \mathfrak{A}_i} P$, each A_i is p-semi-simple. The proof is complete.

We can prove Theorem 8 by a similar argument.

We shall discuss the structure space $\mathfrak{S}$ of all strongly irreducible ideals of A in a later paper.

References

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