122. Fourier Series. III. Wiener's Problem

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1. N. Wiener [1] proposed to study the convergence of the series

$$\sum_{n=1}^{\infty} |s_n(x)-f(x)|^{\lambda}, \tag{1}$$

where $s_n(x)$ is the nth partial sum of the Fourier series of $f(x)$. In the former paper [2], we have proved the following

Theorem 1. *Let $p\geqq\lambda>1$ and $\varepsilon>0$. If*

$$\omega_p(t,f)=\max_{0<u<t}\left(\int_0^{2\pi}|f(x+u)-f(x)|^p dx\right)^{1/p}=O\left(\frac{t^{1/\lambda}}{(\log 1/t)^{(1+\varepsilon)/\lambda}}\right),$$

then the series (1) converges almost everywhere.

Further M. Kinukawa [3] proved the following

Theorem 2. *If one of the following conditions (a), (b), (c) is satisfied, then the series (1) converges almost everywhere:*

(a) $\displaystyle\sum_{k=1}^{\infty} k^{\gamma}(2^{k/\lambda}\omega_p(1/2^k))^p<\infty \qquad (2\geqq p>\lambda>1,\ \gamma>p/\lambda-1),$

(b) $\displaystyle\sum_{k=1}^{\infty} 2^{k}(\omega_p(1/2^k))^p<\infty \qquad (2>p=\lambda>1),$

(c) $\displaystyle\sum_{k=1}^{\infty} k2^{k}(\omega_p(1/2^k))^p<\infty \qquad (p=\lambda=2).$

If $1<p\leqq 2$, Theorem 2 contains Theorem 1 as a particular case. We shall here prove the following

Theorem 3. *If*

$$\sum_{n=1}^{\infty}\omega_\lambda^\lambda(1/n)<\infty,$$

then the series (1) converges almost everywhere.

This theorem contains Theorems 1 and 2, (b) and (c). The method of the proof is that used to proved Theorem 1.

2. **Proof of Theorem 3.** We use a lemma due to A. Zygmund.

Lemma. *Let $p>1$. If*

$$\Big\|\sum_{\nu=m}^{n}\gamma_\nu e^{i\nu x}\Big\|_p\leqq C,$$

$$|\lambda_\nu|<M, \qquad \sum_{\nu=m}^{n-1}|\lambda_\nu-\lambda_{\nu+1}|\leqq M,$$

then

$$\Big\|\sum_{\nu=m}^{n}\gamma_\nu\lambda_\nu e^{i\nu x}\Big\|_p\leqq A_pMC.$$

Let us now prove Theorem 3. It is sufficient to prove

$$\sum_{n=1}^{\infty}\int_0^{2\pi}|s_n(x)-f(x)|^{\lambda}dx<\infty.$$

We have

$$\int_0^{2\pi} |s_n(x)-f(x)|^\lambda dx = \|s_n - f\|_\lambda^\lambda.$$

For the sake of simplicity we suppose

$$f(x) \sim \sum_{\nu=0}^{\infty} c_\nu e^{i\nu x},$$

then

$$f(x+t)-f(x-t) \sim 2\sum_{\nu=0}^{\infty} c_\nu \sin \nu t e^{i\nu x}.$$

By the F. Riesz theorem

$$\|\sum_{\nu=2^k}^{2^{k+1}} c_\nu \sin \nu t e^{i\nu x}\|_\lambda \leqq A_p \omega_\lambda(t, f).$$

Taking $\lambda_\nu = 1/\sin \nu t$, $t=\pi/2^{k+2}$, we get by Lemma

$$\|\sum_{\nu=m}^{n} c_\nu e^{i\nu x}\|_\lambda \leqq A_p \omega_\lambda(\pi/2^{k+2}, f)$$

for $2^k \leqq m < n \leqq 2^{k+1}$. Hence, if $2^k \leqq n < 2^{k+1}$,

$$\|s_n - f\|_\lambda \leqq \|\sum_{\nu=n}^{2^{k+1}} c_\nu e^{i\nu x}\|_\lambda + \sum_{j=k+1}^{\infty} \|\sum_{\nu=2^j}^{2^{j+1}} c_\nu e^{i\nu x}\|_\lambda \leqq A \sum_{j=k+2}^{\infty} \omega_\lambda(\pi/2^j, f)$$

and then we have

$$\sum_{n=1}^{\infty} \|s_n - f\|_\lambda^\lambda \leqq \sum_{n=1}^{\infty}\Big(\sum_{j=k+2}^{\infty} \omega_\lambda(\pi/2^j, f)\Big)^\lambda = \sum_{k=1}^{\infty} 2^k \Big(\sum_{j=k+2}^{\infty} \omega_\lambda(\pi/2^j, f)\Big)^\lambda$$

which is convergent when and only when

$$(2) \qquad \sum_{k=1}^{\infty}\Big(\sum_{n=k}^{\infty} \frac{1}{n} \omega_\lambda(\pi/n, f)\Big)^\lambda$$

is convergent. The inner sum is less than

$$\Big(\sum_{n=k}^{\infty} \frac{\omega_\lambda(\pi/n)^\lambda}{n^{a\lambda}}\Big)^{1/\lambda}\Big(\sum_{n=k}^{\infty} \frac{1}{n^{b\lambda'}}\Big)^{1/\lambda'},$$

where $a>0$, $b>0$, $a+b=1$, $1/\lambda+1/\lambda'=1$. If $1>b>(\lambda-1)/\lambda$, then $\sum_{n=k}^{\infty} \frac{1}{n^{b\lambda'}}$ converges and is less than

$$\Big(\sum_{n=k}^{\infty} \frac{1}{n^{b\lambda'}}\Big)^{1/\lambda'} \leqq \frac{A}{k^{(b\lambda'-1)/\lambda'}}.$$

Thus (2) is less than

$$A \sum_{k=1}^{\infty} \frac{1}{k^{\lambda(b-1)+1}} \sum_{n=k}^{\infty} \frac{\omega_\lambda^\lambda(1/n)}{n^{(1-b)\lambda}} \leqq A \sum_{n=1}^{\infty} \frac{\omega_\lambda^\lambda(1/n)}{n^{(1-b)\lambda}} \sum_{k=1}^{n} \frac{1}{k^{\lambda(b-1)+1}}.$$

Since $0<\lambda(b-1)<1$, the last sum is less than

$$A \sum_{n=1}^{\infty} \frac{\omega_\lambda^\lambda(1/n)}{n^{(1-b)\lambda}} n^{(1-b)\lambda} = A \sum_{n=1}^{\infty} \omega_\lambda^\lambda(1/n).$$

Thus the theorem is proved.

References

[1] N. Wiener: Tauberian theorems, Ann. Math., **31** (1932).
[2] S. Izumi: Some trigonometrical series. XX, Proc. Japan Acad., **32** (1956).
[3] M. Kinukawa: Strong convergence of Fourier series, Proc. Japan Acad., **32** (1956).