# 153. *Remarks on the Sequence of Quasi-Conformal Mappings*

By Kêichi SHIBATA

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1. It seems to me that there are essentially two kinds of definition, stronger and weaker, for quasi-conformal mapping with bounded dilatation. The former is rather classical definition of Grötzsch, Teichmüller and other authors. In 1951 Pfluger suggested the latter [6], and Ahlfors remarkably improved the theory of quasi-conformal mapping by making use of it in recent few years [1–3]. The present Note, which I owe much to the investigations of Ahlfors, is concerned with relations between these definitions.

Definition 1. A topological mapping $w=f(z)$ from a domain $D$ in the $z(=x+iy)$-plane to a domain $\Delta$ in the $w(=u+iv)$-plane is called $K$-QC mapping in $D$, when it satisfies the following conditions there:

I) all the partial derivatives $u_x, u_y, v_x, v_y$ exist and are continuous,

II) $$J(z)=u_xv_y-u_yv_x>0,$$

III) $$\frac{|p|+|q|}{|p|-|q|}\leq K<\infty,$$

where $p, q$ are the complex derivatives of $f$

$$p(z)=f_z=\frac{1}{2}[(u_x+v_y)+i(v_x-u_y)],$$

$$q(z)=f_{\bar{z}}=\frac{1}{2}[(u_x-v_y)+i(v_x+u_y)],$$

and $K$ is a constant $\geq 1$.

Let $\Omega$ be a Jordan domain, on whose boundary four ordered points $z_1, z_2, z_3, z_4$, are marked in the positive sense with respect to $\Omega$. This configuration is named *quadrilateral* and is denoted by $\Omega(z_1, z_2, z_3, z_4)$ or simply by $\Omega$. If one maps a quadrilateral $\Omega$ by means of a sense-preserving homeomorphism $T(z)$, the image $T(\Omega)$ is again a quadrilateral. $\Omega$ can be mapped conformally onto the interior of a rectangle $0<\xi<1$, $0<\eta<\lambda$ in the $\zeta(=\xi+i\eta)$-plane, so that the points $z_1, z_2, z_3, z_4$ correspond to $\zeta=0, 1, 1+i\lambda, i\lambda$ respectively. By *module* of the quadrilateral $\Omega(z_1, z_2, z_3, z_4)$ is meant the positive number $\lambda$, which shall be denoted by $\mathrm{mod}\,\Omega(z_1, z_2, z_3, z_4)$.

Definition 2. A topological mapping $w=f(z)$ which transforms a plane domain $D$ onto another such $\Delta$ is called a $K$-QC* mapping, when it satisfies the following conditions:

I′) the mapping $w=f(z)$ is sense-preserving,

II′) for any quadrilateral $\Omega(z_1, z_2, z_3, z_4)$ contained together with its boundary in $D$ the inequality

$$\operatorname{mod} f(\Omega(z_1, z_2, z_3, z_4)) \leq K \operatorname{mod} \Omega(z_1, z_2, z_3, z_4)$$

holds, where $K$ is a finite constant $\geq 1$.

We easily see by the module theorem that $K$-QC mapping is $K$-QC*.

2. Lemma 1. *Suppose that a sequence $\{C_n\}$ of Jordan curves converges to a Jordan curve $C$ containing the origin $w=0$ in its interior in Fréchet sense and that the finite domain $D_n$ bounded by each curve $C_n$ also contains $w=0$. Let $w=F_n(z)$ ($F_n(0)=0$, $F_n'(0)>0$) be the function which maps the unit disk $|z|<1$ conformally onto $D_n$. Then the sequence $\{F_n(z)\}$ of the mapping functions converges uniformly on $|z|=1$* [4].

Proof. i) $\{F_n(z)\}$ is equicontinuous on $|z|=1$. For otherwise, there would exist two sequences $\{z_k'\}$, $\{z_k''\}$ on $|z|=1$, a subsequence $\{F_{n_k}(z)\}$ of $\{F_n(z)\}$ and a positive number $\delta_1$, such that we have

$$|F_{n_k}(z_k')-F_{n_k}(z_k'')| \geq \delta_1 > 0 \ (k=1,2,\cdots), \ \lim_{k\to\infty} |z_k'-z_k''|=0.$$

Without loss of generality we may assume $z_k' \to z_0$, $z_k'' \to z_0$ for $k\to\infty$. We can choose parametrizations $w_n(t)$, $w(t)$ $(0\leq t\leq 1)$ of the Jordan curves $C_n$ $(n=1,2,\cdots)$ and $C$, such that $\{w_n(t)\}$ converges to $w(t)$ uniformly on the interval $[0, 1]$. If we put $F_{n_k}(z_k')=w_k'=w_{n_k}(t_k')$, $F_{n_k}(z_k'')=w_k''=w_{n_k}(t_k'')$, then we have $|t_k'-t_k''|\geq\alpha>0$ $(k=1,2,\cdots)$. For arbitrary $\varepsilon>0$ we choose a number $k_0$ so large that the inequalities $|z_0-z_k'|<\varepsilon$, $|z_0-z_k''|<\varepsilon$ simultaneously hold for $k\geq k_0$. The common part of the circle $|z-z_0|=r$ (resp. the disk $|z-z_0|\leq r$) with the unit disk $|z|\leq 1$ will be transformed by $F_{n_k}(z)$ to some cross-cut $\Gamma_{n_k,r}$ (resp. some subdomain $D_{n_k,r}$) of $D_{n_k}$, whose endpoints shall be denoted by $A_{k,r}=w_{n_k}(t_k'(r))$, $B_{k,r}=w_{n_k}(t_k''(r))$. Then $|t_k'(r)-t_k''(r)|>\alpha$ $(\varepsilon\leq r\leq 1;$ $k\geq k_0+1)$. Since $C$ is a Jordan curve, we have $|w(t_k'(r))-w(t_k''(r))| \geq\beta>0$. Consequently we see by our assumption of Fréchet convergence that there exists a positive integer $k_1$ depending on $\gamma<\beta$, such that

$$|w_{n_k}(t_k'(r))-w_{n_k}(t_k''(r))|\geq\gamma>0 \quad \text{for all } r\in[\varepsilon, 1]$$

provided $k\geq k_1$. Thus we can extract a contradiction from the well-known inequality

$$\int_\varepsilon^1 \frac{dr}{r} \leq \frac{2\pi}{\gamma^2}\int_{A(\varepsilon)}^{A(1)} dA(r),$$

where $A(r)$ means the area of $D_{n_k,r}$ $(k\geq k_1)$.

ii) Let $F(z)$ be the function mapping $|z|<1$ conformally onto the interior of $C$ ($F(0)=0$, $F'(0)>0$). Then by Carathéodory's theorem $\{F_n(z)\}$ converges uniformly to $F(z)$ in $|z|<1$. $F(z)$ is continuous on $|z|\leq 1$.

iii) $\{F_n(z)\}$ converges to $F(z)$ uniformly on $|z|=1$. For otherwise, $\max\limits_{|z|=1}|F(z)-F_n(z)|\geq\alpha'>0$ $(n=1,2,\cdots)$. By the maximum-modulus principle and i) the family $\{F_n(z)\}$ is normal on $|z|\leq 1$. Namely, a suitable subsequence $\{F_{n_\nu}(z)\}$ of $\{F_n(z)\}$ can be chosen so that it is uniformly convergent on $|z|\leq 1$. Put

$$\lim_{\nu\to\infty} F_{n_\nu}(z)=F_0(z) \quad \text{on } |z|\leq 1. \tag{1}$$

Then by ii) $F_0(z)\equiv F(z)$ in $|z|<1$ and accordingly on $|z|\leq 1$. Therefore we would have $\max\limits_{|z|=1}|F_0(z)-F_{n_\nu}(z)|\geq\alpha'$ $(\nu=1,2,\cdots)$, which is contrary to (1).

Theorem 1. *Let $\zeta=\varphi_n(z)$ $(\varphi_n(0)=0;\ n=1,2,\cdots)$ be a K-QC mapping from $|z|<1$ to $|\zeta|<1$. If the sequence $\{\varphi_n(z)\}$ converges to a function $\varphi(z)$ uniformly in $|z|<1$, $\zeta=\varphi(z)$ is a K-QC* mapping from $|z|<1$ to $|\zeta|<1$.*

Proof. It is known that $\zeta=\varphi(z)$ supplies a homeomorphism from $|z|<1$ to $|\zeta|<1$ [7]. Let us fix a rectangle $R$ confined with its boundary $B$ in $|z|<1$, whose vertices shall be denoted by $z_1, z_2, z_3, z_4$. We write $\varphi_n(z_k)=\zeta_k^{(n)}$, $\varphi(z_k)=\zeta_k$ $(k=1,2,3,4;\ n=1,2,\cdots)$ for later use. Suppose that $B$ is transformed by $\varphi_n(z)$ to $C_n$ and by $\varphi(z)$ to $C$. Then $C_n$ and $C$ are Jordan curves, and the sequence $\{C_n\}$ converges to $C$ in Fréchet sense. Let $[C_n]$ (resp. $[C]$) be the interior of $C_n$ (resp. $C$). If we put $\varphi(z_0)=\zeta_0$ for the centre $z_0$ of $R$, $\zeta_0$ will be contained in $[C_n]$ from some number $N$ onwards. Let $\zeta=G_n(Z)$ (resp. $\zeta=G(Z)$) be the function which maps $[C_n]$ (resp. $[C]$) conformally onto $|Z|<1$ with the normalization $G_n(0)=G(0)=\zeta_0$, $G_n'(0)>0$, $G'(0)>0$; $n\geq N$. If we put $Z_k^{(n)}=G_n^{-1}(\zeta_k^{(n)})$, $Z_k=G^{-1}(\zeta_k)$, we see at once

$$\lim_{n\to\infty} G_n(Z_k^{(n)})=\lim_{n\to\infty}\zeta_k^{(n)}=\zeta_k=G(Z_k). \tag{2}$$

Now, if $\{Z_k^{(n)}\}$ never tend to $Z_k$ for $n\to\infty$, then for a suitable subsequence, say again $\{Z_k^{(n)}\}$, we would have $\lim\limits_{n\to\infty} Z_k^{(n)}=Z_k'\neq Z_k$. Therefore $\lim\limits_{n\to\infty} G_n(Z_k^{(n)})=G(Z_k')=G(Z_k)$ by Lemma 1 and (2). We must have $Z_k'=Z_k$ $(k=1,2,3,4)$, since $G(Z)$ is univalent. We conclude

$$\lim_{n\to\infty}\operatorname{mod}\Gamma(Z_1^{(n)}, Z_2^{(n)}, Z_3^{(n)}, Z_4^{(n)})=\operatorname{mod}\Gamma(Z_1, Z_2, Z_3, Z_4),$$

where $\Gamma$ denotes the unit disk. It is equivalent to the relation

$$\lim_{n\to\infty}\operatorname{mod}[C_n](\zeta_1^{(n)}, \zeta_2^{(n)}, \zeta_3^{(n)}, \zeta_4^{(n)})=\operatorname{mod}[C](\zeta_1, \zeta_2, \zeta_3, \zeta_4),$$

from which our desired inequality

$$\operatorname{mod}[C](\zeta_1, \zeta_2, \zeta_3, \zeta_4)\leq K\operatorname{mod} R(z_1, z_2, z_3, z_4)$$

follows.

3. The following propositions will play fundamental rôle throughout the whole theory of $K$-QC* mapping.

Let $w=f(z)$ be a $K$-QC* mapping defined in a rectangle $R$: $a<x<b$, $c<y<d$. Then

1° $f(z)$ is totally differentiable at almost all points of $R$

$$df(z)=p(z)dz+q(z)d\bar{z};$$

2° at such a point there hold the inequalities

$$|p|^2=|q|^2\geq 0,\quad (|p|+|q|)^2\leq K(|p|^2-|q|^2);$$

3° for almost every value of $y_0$ belonging to the interval $(c, d)$ $f(x, y_0)$ is absolutely continuous with respect to $x$ in the interval $(a, b)$ [3, 5].

The next is due to Ahlfors [3]:

4° any set of 2-dimensional measure zero in the $z$-plane is transformed by $w=f(z)$ to a set of 2-dimensional measure zero in the $w$-plane.

It follows from 1°, 2° and 4° that $p(z)\neq 0$ a.e. in $R$, whence the measurable function $h(z)=q(z)/p(z)$ is defined a.e. in $R$ and satisfies

$$|h(z)|\leq\frac{K-1}{K+1}<1 \qquad \text{a.e. in } R.$$

Lemma 2. *Let $\zeta=\varphi_n(z)$ $(\varphi_n(0)=0,\ \varphi_n(1)=1)$ be a K-QC* mapping from $|z|<1$ to $|\zeta|<1$, and let us write*

$$d\varphi_n=p_n dz+q_n d\bar{z},\qquad h_n(z)=\frac{q_n(z)}{p_n(z)}.$$

*If $\lim_{n\to\infty}\iint_{|z|<1}|h_n(z)|^2 dxdy=0$, then we have $\lim_{n\to\infty}\varphi_n(z)=z$ uniformly on $|z|\leq 1$.*

Proof. Let $C$ be an arbitrary rectifiable Jordan curve in $|z|<1$ and let $[C]$ be its interior. Then by 2°, 3° and Schwarz's inequality

$$\left|\int_C\varphi_n(z)dz\right|^2=4\left|\iint_{[C]}q_n(z)dxdy\right|^2=4\left|\iint_{[C]}h_n(z)p_n(z)dxdy\right|^2$$

$$\leq 4\iint_{[C]}|h_n(z)|^2 dxdy\iint_{[C]}|p_n(z)|^2 dxdy\leq 4K\pi\iint_{[C]}|h_n(z)|^2 dxdy.$$

Since the sequence $\{\varphi_n(z)\}$ forms a normal family on $|z|\leq 1$ [1], its suitable subsequence $\{\varphi_{n_\nu}(z)\}$ will be uniformly convergent there. If we put

$$\lim_{\nu\to\infty}\varphi_{n_\nu}(z)=\varphi(z)\qquad |z|\leq 1,$$

we have by the above inequality

$$\int_C\varphi(z)dz=\lim_{\nu\to\infty}\int_C\varphi_{n_\nu}(z)dz=0.$$

Therefore $\zeta=\varphi(z)$ must be regular in $|z|<1$, while it is a topological mapping from $|z|<1$ to $|\zeta|<1$ by Theorem 1. Thus $\varphi(z)\equiv z$. If the original sequence $\{\varphi_n(z)\}$ do not converge to $z$ uniformly on $|z|\leq 1$, we would have for a suitable subsequence $\{\varphi_{n_k}(z)\}$

$$\max_{|z|\leq 1}|z-\varphi_{n_k}(z)|\geq a>0\quad (k=1,2,\cdots).$$

This is a contradiction, since $\{\varphi_{n_k}(z)\}$ always contains a subsequence converging uniformly to $z$ on $|z|\leq 1$.

Lemma 3. *For any function $S(z)$ of summable square it is possible* *to choose a sequence* $\{S_n(z)\}$ $(n=1,2,\cdots)$ *of functions* $C^1$ *which vanish outside a compact set, so that*

$$\lim_{n\to\infty}\iint|S(z)-S_n(z)|^2dxdy=0.$$

Proof. Given any $\varepsilon>0$, we can find a bounded measurable function $s_\varepsilon(z)$ vanishing outside a compact set, such that

$$\iint|S(z)-s_\varepsilon(z)|^2dxdy<\frac{\varepsilon}{3}.$$

Let $s_{\varepsilon,m}(z)$ be the arithmetic mean of the function $s_\varepsilon(z)$ over the disk $|\zeta-z|\leq 1/m$ $(m=1,2,\cdots)$

$$s_{\varepsilon,m}(z)=\frac{m^2}{\pi}\int_0^{1/m}\int_0^{2\pi}s_\varepsilon(z+re^{i\theta})rdrd\theta.$$

Then $s_{\varepsilon,m}(z)$ is continuous and uniformly (with respect to $m$) bounded function, and

$$\lim_{m\to\infty}s_{\varepsilon,m}(z)=s_\varepsilon(z) \qquad \text{a.e.}$$

Therefore there exists a number $m_0(\varepsilon)$, such that for $m\geq m_0(\varepsilon)$ we have

$$\iint|s_\varepsilon(z)-s_{\varepsilon,m}(z)|^2dxdy<\frac{\varepsilon}{3}.$$

Let us mean $s_{\varepsilon,m}(z)$ arithmetically once more over a disk with radius $1/k$ $(k=1,2,\cdots)$ to obtain the smooth function

$$s_{\varepsilon,m,k}(z)=\frac{k^2}{\pi}\int_0^{1/k}\int_0^{2\pi}s_{\varepsilon,m}(z+re^{i\theta})rdrd\theta.$$

There exists a number $k_0(\varepsilon,m)$, such that for $k\geq k_0(\varepsilon,m)$ we have

$$\iint|s_{\varepsilon,m}(z)-s_{\varepsilon,m,k}(z)|^2dxdy<\frac{\varepsilon}{3}.$$

Consequently there holds the inequality

$$\iint|S(z)-s_{\varepsilon,m,k}(z)|^2dxdy<\varepsilon, \tag{3}$$

so far as $m,k$ is large enough for given $\varepsilon$. Let $S_n(z)$ be one of the functions $s_{\varepsilon,m,k}(z)$ satisfying (3) when $\varepsilon=1/n$. The proof is completed.

Theorem 2. *Given any K-QC* mapping* $\zeta=\varphi(z)$ *from* $|z|<1$ *to* $|\zeta|<1$, *there exists a sequence* $\{\varphi_n(z)\}$ *of functions which converges to* $\varphi(z)$ *uniformly on* $|z|\leq 1$, *such that each function* $\zeta=\varphi_n(z)$ *furnishes a K-QC mapping from* $|z|<1$ *to* $|\zeta|<1$.

Proof. We may assume $\varphi(0)=0$, $\varphi(1)=1$ without loss of generality. Let us write $d\varphi=pdz+qd\bar{z}$, $h=q/p$ a.e. in $|z|<1$ and put $h(z)=0$ where it is not defined. Then we can construct by the method in Lemma 3 a sequence $\{h_n(z)\}$ of continuously differentiable functions which tends to $h(z)$ in $L^2$ sense. Each $h_n(z)$ has a uniformly bounded compact carrier and $|h_n(z)|\leq(K-1)/(K+1)<1$. Ahlfors proved: *for any square-summable and Hölder-continuous function* $h_n(z)$ $(|h_n(z)|$

$\leq \kappa < 1$) *there exists a function* $w = f_n(z) \in C^1$ *which supplies a homeomorphism between the whole z- and w-plane, such that* $\tau_n(z)/\sigma_n(z) = h_n(z)$, where $\sigma_n(z) = \partial f_n/\partial z$, $\tau_n(z) = \partial f_n/\partial \bar{z}$ [2]. Let $\zeta = \Psi_n(w)$ be the function which maps conformally onto $|\zeta| < 1$ the image of $|z| < 1$ by $f_n(z)$ and let $\varphi_n(z)$ be the composite function $\zeta = \varphi_n(z) = \Psi_n(f_n(z))$ with the normalization $\varphi_n(0) = 0$, $\varphi_n(1) = 1$. Every $\varphi_n(z)$ is $K$-QC, and we write $d\varphi_n = p_n dz + q_n d\bar{z}$. One may express the composite function $\varphi_n \circ \varphi^{-1}$ by means of $\tilde{\varphi}_n(\zeta)$ with the independent variable $\zeta$. It is obviously a $K^2$-QC* mapping between the unit disks which can be considered conformal with respect to some Riemannian metric $|d\zeta + \tilde{h}_n(\zeta) d\bar{\zeta}|$. In order to calculate $\tilde{h}_n(\zeta)$, $dz$ and $d\bar{z}$ should be eliminated from three relations

$$d\varphi_n = p_n dz + q_n d\bar{z}, \quad d\varphi = p dz + q d\bar{z}, \quad d\bar{\varphi} = \bar{q} dz + \bar{p} d\bar{z}.$$

We obtain

$$(|p|^2 - |q|^2) d\varphi_n = (p_n \bar{p} - q_n \bar{q}) d\varphi + (p q_n - p_n q) d\bar{\varphi},$$

and finally

$$\tilde{h}_n(\zeta) = \frac{\partial \varphi_n}{\partial \bar{\varphi}} \Big/ \frac{\partial \varphi_n}{\partial \varphi} = \frac{p(z)}{\overline{p(z)}} \frac{h_n(z) - h(z)}{1 - h_n(z)\overline{h(z)}}.$$

Since

$$\left| \frac{h_n(z) - h(z)}{1 - h_n(z)\overline{h(z)}} \right| \leq \frac{K^2 - 1}{2K},$$

it follows by the well-known theorem of Lebesgue that

$$\lim_{n\to\infty} \iint_{|\zeta|<1} |\tilde{h}_n(\zeta)|^2 d\xi d\eta = \iint_{|z|<1} \lim_{n\to\infty} \left| \frac{h_n(z) - h(z)}{1 - h_n(z)\overline{h(z)}} \right|^2 (|p(z)|^2 - |q(z)|^2) dx dy = 0.$$

Therefore by Lemma 2 the sequence $\{\tilde{\varphi}_n(\zeta)\}$ tends uniformly to the identity on $|\zeta| \leq 1$ for $n \to \infty$, in other words, $\lim_{n\to\infty} \varphi_n(z) = \varphi(z)$ uniformly on $|z| \leq 1$.

## References


[1] Ahlfors, L. V.,: On quasiconformal mappings, Journ. d'anal. Math., **3**, 1–58, 207–208 (1953–54).

[2] ——: Conformality with respect to Riemannian metrics, Ann. Acad. Sci. Fenn., A,I, **206**, 1–22 (1955).

[3] ——: On quasiconformal mappings, Lecture at Osaka University, February, 1956.

[4] Courant, R.,: Über eine Eigenschaft der Abbildungsfunktion bei konformer Abbildung, Nachr. Ges. Wiss. Göttingen, 69–70 (1922).

[5] Mori, A.,: On quasi-conformality and pseudo-analyticity (in Japanese), Sūgaku, **7**, 75–89 (1955).

[6] Pfluger, A.,: Quasikonforme Abbildungen und logarithmische Kapazität, Ann. Inst. Fourier, **2**, 69–80 (1951).

[7] Tôki, Y., and Shibata, K.,: On the pseudo-analytic functions, Osaka Math. J., **6**, 145–165 (1954).