152. *Note on Free Algebraic Systems*

By Tsuyoshi FUJIWARA
Department of Mathematics, Yamaguchi University
(Comm. by K. SHODA, M.J.A., Nov. 12, 1956)

In his paper,[1] K. Shoda has defined only the free A-algebraic systems, when he has discussed the free algebraic systems. However, in this note, we shall define free algebraic systems more generally. And we shall show a generalization of the Shoda's fundamental theorem[2] (Theorems 1, 2 and 3), and a necessary and sufficient condition for the existence of the free algebraic system with an arbitrary set of relations (Theorem 3). Finally, we shall show a characterization of the algebraic systems defined by only a set of relations, i.e. the A-algebraic systems satisfying a set of relations (Theorem 4).

Throughout this note, the system V of single-valued compositions will be fixed. Let E be a set of generators, then the absolutely free algebraic system $F(E,\phi)$[3] is defined. And let P be a family of postulates with respect to V and E, then P-algebraic systems generated by E are defined as residue class systems of $F(E,\phi)$ satisfying P. And (E,P) denotes the set of all P-algebraic systems generated by E. Moreover, let R be a set of relations (identities) in $F(E,\phi)$, then the P-algebraic systems satisfying R generated by E are defined. And (E,P,R) denotes the set of all such P-algebraic systems.

An algebraic system $\mathfrak{F}$ is called a free P-algebraic system with a set R generated by E, or a free algebraic system belonging to (E,P,R), when $\mathfrak{F}$ is contained in (E,P,R) and every algebraic system in (E,P,R) is a residue class system of $\mathfrak{F}$. And we denote it by $F(E,P,R)$.

Theorem 1. *If an algebraic system $\mathfrak{A}$ is contained in (E,P,R), then there exists a set S of relations satisfying $\mathfrak{A}=F(E,P,S)$ and $S\supseteq R$.*

Proof. Let $\mathfrak{A}\in(E,P,R)$, then it is clear that $\mathfrak{A}\in(E,\phi,R)$.[4] Hence there exists a set S of relations satisfying $\mathfrak{A}=F(E,\phi,S)$ and $S\supseteq R$ by

1) K. Shoda: Allgemeine Algebra, Osaka Math. J., **1** (1949).

2) Using our notations, we can show the Shoda's fundamental theorem for the free algebraic systems as follows: Let P be a family of composition-identities. Then i) there exists a free algebraic system $F(E,P,R)$ for every set R of relations, ii) if an algebraic system $\mathfrak{A}$ is contained in (E,P,R), then there exists a set S of relations satisfying $\mathfrak{A}=F(E,P,S)$ and $S\supseteq R$, and iii) if $R\subseteq S$, then $F(E,P,S)$ is a residue class system of $F(E,P,R)$.

3) In his paper 1), K. Shoda has denoted by $O(E)$ the absolutely free algebraic system.

4) ϕ denotes the empty set.

the Shoda's fundamental theorem. Now, it is verified that $\mathfrak{A}\in(E,P,S)$, because $\mathfrak{A}\in(E,P,R)$ and $\mathfrak{A}\in(E,\phi,S)$. Moreover, any algebraic system in (E,P,S) is a residue class system of $\mathfrak{A}$, since $\mathfrak{A}=F(E,\phi,S)\in(E,P,S)\subseteq(E,\phi,S)$. Hence $\mathfrak{A}$ is a free algebraic system belonging to (E,P,S), i.e. $\mathfrak{A}=F(E,P,S)$.

Theorem 2. *Suppose that there exists $F(E,P,R)$ for every set R of relations. If $R\subseteq S$, then $F(E,P,S)$ is a residue class system of $F(E,P,R)$.*

Proof. Let $R\subseteq S$, then it is clear that $(E,P,S)\subseteq(E,P,R)$. Hence $F(E,P,S)\in(E,P,R)$. Therefore $F(E,P,S)$ is a residue class system of $F(E,P,R)$.

Let R and S be two sets of relations in $F(E,\phi)$. A condition that R implies S is called an implication from R to S, or simply an implication. Moreover, a family P of postulates is said to be equivalent to P^*, when $(E,P)=(E,P^*)$.

Theorem 3. *In order that there exists $F(E,P,R)$ for every set R of relations, it is necessary and sufficient that P is equivalent to a family P^* of implications.*

(I) Proof of sufficiency: Let P be equivalent to a family P^* of implications, then it is obvious that $(E,P,R)=(E,P^*,R)$ for every set R of relations. Hence it is sufficient to prove the existence of $F(E,P^*,R)$ for any set R of relations.

Let $\mathfrak{A}$ be any algebraic system contained in (E,P^*,R). It is, of course, clear that $\mathfrak{A}\in(E,\phi)$. Hence there exists a congruence $\theta_{\mathfrak{A}}$ of $F(E,\phi)$ satisfying $\mathfrak{A}=F(E,\phi)/\theta_{\mathfrak{A}}$.[5] Now, let $\varphi_R=\bigcap_{\mathfrak{A}\in(E,P^*,R)}\theta_{\mathfrak{A}}$, then it is verified that $F(E,\phi)/\varphi_R\in(E,P^*,R)$, since P^* is a family of implications. And it is clear that any algebraic system $\mathfrak{A}$ in (E,P^*,R) is a residue class system of $F(E,\phi)/\varphi_R$, since $\mathfrak{A}=F(E,\phi)/\theta_{\mathfrak{A}}$ and $\theta_{\mathfrak{A}}\geqq\varphi_R$. Hence $F(E,\phi)/\varphi_R$ is a free algebraic system belonging to (E,P^*,R), i.e. there exists a free algebraic system $F(E,P^*,R)=F(E,\phi)/\varphi_R$.

(II) Proof of necessity: Suppose that there exists $F(E,P,R)$ for every set R of relations. First we shall define a set $\overline{R}$, for every set R, as the set-sum of all sets S satisfying $F(E,P,R)=F(E,P,S)$. Then it is verified that

$$(*)\qquad F(E,P,R)=F(E,P,\overline{R})=F(E,\phi,\overline{R})$$

for every set R of relations. Now we shall define P^* as the family of all the implications written in the form that R implies $\overline{R}$. In the following, we shall show the fact that P is equivalent to P^*, i.e. $(E,P)=(E,P^*)$.

Let $\mathfrak{A}\in(E,P)$, then there exists a set R of relations satisfying

5) $F(E,\phi)/\theta_{\mathfrak{A}}$ denotes the residue class system of $F(E,\phi)$ modulo $\theta_{\mathfrak{A}}$.

$\mathfrak{A}=F(E,P,R)$ by Theorem 1. And by $(*)$ we get that $\mathfrak{A}=(E,P,R)$ $=F(E,\phi,\overline{R})$. Now it is verified that $S\subseteq\overline{R}$ implies $\overline{S}\subseteq\overline{R}$ by Theorem 2. Hence $F(E,\phi,\overline{R})$ satisfies the family P^* of implications. Therefore, $(E,P)\subseteq(E,P^*)$.

Hereafter we shall prove that $(E,P)\supseteq(E,P^*)$. It has been verified in (I) that there exists $F(E,P^*,R)$ for every set R of relations. Hence we can define $\tilde{R}$, for every set R, as the set-sum of all sets S satisfying $F(E,P^*,R)=F(E,P^*,S)$. Now let $\mathfrak{A}\in(E,P^*)$, then there exists a set R of relations satisfying $\mathfrak{A}=F(E,P^*,R)$. And it is clear that $\mathfrak{A}=F(E,P^*,R)=F(E,\phi,\tilde{R})$. Now it is evident that $\tilde{R}\subseteq\overline{\overline{\tilde{R}}}$. And it is verified that $\tilde{R}\supseteq\overline{\overline{\tilde{R}}}$, since $\mathfrak{A}$ satisfies the family P^* containing the implication from R to $\overline{\overline{\tilde{R}}}$. Hence $\tilde{R}=\overline{\overline{\tilde{R}}}$ and $\mathfrak{A}=F(E,\phi,\tilde{R})$ $=F(E,\phi,\overline{\overline{\tilde{R}}})$. Moreover, $F(E,\phi,\overline{\overline{\tilde{R}}})=F(E,P,\overline{\overline{\tilde{R}}})$ by $(*)$. Therefore, we get that $\mathfrak{A}\in(E,P)$, $(E,P^*)\subseteq(E,P)$, and hence $(E,P)=(E,P^*)$.

Theorem 4. *In order that (i) there exists $F(E,P,R)$ for every set R of relations, and (ii) any residue class system of an algebraic system in (E,P) is contained in (E,P), it is necessary and sufficient that P is equivalent to a family P^* of relations.*

Proof. The sufficient part of this theorem is evident. Hereafter we shall prove the necessary part. Suppose that P satisfies the conditions (i) and (ii). Then there exists $F(E,P,\phi)$. And $F(E,P,\phi)\in$ $(E,P)\subseteq(E,\phi)$. Hence there exists a set P^* of relations satisfying $F(E,\phi,P^*)=F(E,P,\phi)$, i.e. $F(E,P^*,\phi)=F(E,P,\phi)$. Now we shall show that P is equivalent to P^*, i.e. $(E,P)=(E,P^*)$. Let $\mathfrak{A}\in(E,P^*)$, then there exists a congruence θ of $F(E,P^*,\phi)$ satisfying $\mathfrak{A}=F(E,P^*,\phi)/\theta$. And clearly $\mathfrak{A}=F(E,P,\phi)/\theta$. Hence $\mathfrak{A}\in(E,P)$, and $(E,P^*)\subseteq(E,P)$. The converse $(E,P)\subseteq(E,P^*)$ is similarly obtained as mentioned above. Hence $(E,P)=(E,P^*)$.