151. Lebesgue's Constant of (R, λ, k) Summation

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(Comm. by Z. SUETUNA, M.J.A., Nov. 12, 1956)

1. Suppose that $\lambda(w)=\exp\mu(w)$ satisfies the following conditions:
(i) $\mu(w)$ is differentiable and monotone increasing in $(0,\infty)$ and $\mu(w)\to\infty$ as $w\to\infty$.
(ii) $\mu'(w)$ is monotone decreasing for $w>A$, and $\mu'(w)\to 0$, $w\mu'(w)\to\infty$ as $w\to\infty$.
(iii) $\lambda'(w)$ increases monotonously for $w>A$.

We shall prove the following

Theorem. *If we denote by $L_R(w)$ the Lebesgue constant of the $(R, \lambda(w), k)$ summation, $k>0$, then we have*

$$L_R(w)\cong \frac{4}{\pi^2}\log\{\mu'(w)w\}.$$

From this theorem, we can see that there is a continuous function which is not $(R, \lambda(w), k)$ summable, when $\lambda(w)$ satisfies the above conditions.[1]

2. **Proof.** As usual we put $\varphi(t)=\{f(x+t)+f(x-t)-2f(x)\}/2$ and write

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}A_n(x).$$

Then the $[w]$th partial sum is

$$c_0(w)=\sum_{n<w}A_n(x)=S_{[w]}(x)=\frac{2}{\pi}\int_0^{\pi}\varphi(t)\frac{\sin([w]+1/2)t}{2\sin(1/2)t}dt.$$

Hence for $k>0$ and $w>0$, the Riesz sum of type $\lambda(w)$ and of order k is

$$\begin{aligned}C_k(w)&=\sum_{n<w}\{\lambda(w)-\lambda(n)\}^kA_n(x)=k\int_0^w\{\lambda(w)-\lambda(x)\}^{k-1}\lambda'(x)c_0(x)dx\\&=\frac{2k}{\pi}\int_0^w\{\lambda(w)-\lambda(x)\}^{k-1}\lambda'(x)dx\int_0^{\pi}\varphi(t)\frac{\sin([x]+1/2)t}{2\sin(1/2)t}dt\\&=\frac{2}{\pi}\int_0^{\pi}\varphi(t)\frac{k}{2\sin(1/2)t}dt\int_0^w\{\lambda(w)-\lambda(x)\}^{k-1}\lambda'(x)\sin([x]+1/2)t\,dx,\end{aligned}$$

and then the Riesz mean is

$$\frac{C_k(w)}{\{\lambda(w)\}^k}=\frac{2}{\pi}\int_0^{\pi}\varphi(t)\frac{k}{\{\lambda(w)\}^k2\sin(1/2)t}dt\int_0^w\{\lambda(w)-\lambda(x)\}^{k-1}\lambda'(x)\sin([x]+1/2)t\,dx.$$

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1) This was communicated at the Annual Meeting of the Mathematical Society of Japan, in May, 1956.

Thus the Fourier kernel of Riesz's method of summation becomes

$$K(w, t)=\frac{k}{\{\lambda(w)\}^k 2\sin(1/2)t}\int_0^w \{\lambda(w)-\lambda(x)\}^{k-1}\lambda'(x)\sin([x]+1/2)t\,dx,$$

hence the Lebesgue constant $L_R(w)$ for $(R, \lambda(w), k)$ is

$$L_R(w)=\frac{2}{\pi}\int_0^\pi \left| \frac{k}{\{\lambda(w)\}^k 2\sin(1/2)t} \int_0^w \{\lambda(w)-\lambda(x)\}^{k-1}\lambda'(x)\sin([x]+1/2)t\,dx\right| dt.$$

Let us write

$$\int_0^w \{\lambda(w)-\lambda(x)\}^{k-1}\lambda'(x)\sin([x]+1/2)t\,dx=\int_0^A+\int_A^w=I_1+I_2,$$

where A is a positive constant. Then, for $k>0$, $w>A$, we get

$$|I_1|\leqq\int_0^A \{\lambda(w)-\lambda(x)\}^{k-1}\lambda'(x)\,dx=O(\{\lambda(w)\}^{k-1}).$$

When $k\geqq 1$, since $\lambda(x)$ and $\lambda'(x)$ are monotone increasing, we get, by the second mean value theorem,

$$I_2=\{\lambda(w)-\lambda(A)\}^{k-1}\lambda'(w)\int_{w'}^{w''}\sin([x]+1/2)t\,dx \quad (A\leqq w'<w''\leqq w)$$

$$=O(\{\lambda(w)-\lambda(A)\}^{k-1}\lambda'(w))O(t^{-1})=O(\{\lambda(w)\}^k\mu'(w)t^{-1}).$$

When $0<k<1$

$$I_2=\int_A^{w-\frac{1}{t}}+\int_{A-\frac{1}{t}}^{w}=I_{21}+I_{22}, \text{ say.}$$

We get by the second mean value theorem

$$I_{21}=\{\lambda(w)-\lambda(w-1/t)\}^{k-1}\lambda'(w-1/t)\int_A^\xi \sin([x]+1/2)t\,dx \quad (A\leqq\xi\leqq w-1/t)$$

$$=\{t^{-1}\lambda'(\eta)\}^{k-1}\lambda'(w-1/t)O(t^{-1})=O(\{\lambda'(\eta)\}^k t^{-k}) \quad (w-1/t\leqq\eta\leqq w)$$

$$=O(\{\lambda'(w)\}^k t^{-k})=O(\{\lambda(w)\}^k\{\mu'(w)\}^k t^{-k}).$$

$$I_{22}=\int_{w-\frac{1}{t}}^{w}\{\lambda(w)-\lambda(x)\}^{k-1}\lambda'(x)O(1)dx=O([\{\lambda(w)-\lambda(x)\}^k]_{w-\frac{1}{t}}^{w})$$

$$=O(\{\lambda(w)-\lambda(w-1/t)\}^k)=O(t^{-1}\lambda'(\eta))^k=O(\{\lambda(w)\}^k\{\mu'(w)\}^k t^{-k}).$$

Thus the integrand of $L_R(w)$ is

$$(1) \qquad O\left(\frac{k}{2\sin(1/2)t}\left\{\frac{1}{\lambda(w)}+\mu'(w)t^{-1}\right\}\right), \quad (k\geqq 1)$$

and is

$$(2) \qquad O\left(\frac{k}{2\sin(1/2)t}\left[\frac{1}{\lambda(w)}+\{\mu'(w)\}^k t^{-k}\right]\right), \quad (1>k>0).$$

On the other hand, for $k>0$,

$$\int_0^w \{\lambda(w)-\lambda(x)\}^{k-1}\lambda'(x)\sin([x]+1/2)t\,dx$$

$$=\int_0^w \{\lambda(w)-\lambda(x)\}^{k-1}\lambda'(x)\sin xt\,dx$$

$$+\int_0^w \{\lambda(w)-\lambda(x)\}^{k-1}\lambda'(x)\{\sin([x]+1/2)t-\sin xt\}\,dx$$

$$=\frac{1}{k}\int_0^w t\{\lambda(w)-\lambda(x)\}^k\cos xt\,dx$$

$$+\int_0^w \{\lambda(w)-\lambda(x)\}^{k-1}\lambda'(x)\,2\cos\frac{[x]+(1/2)+x}{2}\sin\frac{[x]+(1/2)-x}{2}\,dx$$

$$=\frac{1}{k}\int_0^w t\{\lambda(w)-\lambda(x)\}^k\cos xt\,dx+O(\sin(3/4)t)\int_0^w \{\lambda(w)-\lambda(x)\}^{k-1}\lambda'(x)dx$$

$$=\frac{1}{k}\int_0^w t\{\lambda(w)\}^k\Big[1+\sum_{q=1}^{\infty}(-1)^q\binom{k}{q}\Big\{\frac{\lambda(x)}{\lambda(w)}\Big\}^q\Big]\cos xt\,dx+O(\{\lambda(w)\}^k\sin(3/4)t)$$

$$=\frac{1}{k}\Big\{\lambda(w)\Big\}^k\frac{t}{t}\sin wt+O(t)\Big[\sum_{q=1}^{\infty}\Big|\binom{k}{q}\Big|\{\lambda(w)\}^{k-q}\int_0^w\{\lambda(x)\}^q\,dx\Big]$$

$$+O(\{\lambda(w)\}^k\sin(3/4)t).$$

Since [1]

$$\int_0^w\{\lambda(x)\}^q\,dx=O\Big(\frac{1}{\mu'(w)}\Big)\int_0^w\mu'(x)\{\exp\mu(x)\}^q\,dx=O\Big(\frac{\{\lambda(w)\}^q}{q\mu'(w)}\Big),$$

we get $\displaystyle\sum_{q=1}^{\infty}\Big|\binom{k}{q}\Big|\{\lambda(w)\}^{k-q}\int_0^w\{\lambda(w)\}^q\,dx=O\Big(\sum_{q=1}^{\infty}\Big|\binom{k}{q}\Big|\{\lambda(w)\}^{k-q}\frac{\{\lambda(w)\}^q}{q\mu'(w)}\Big)$

$$=\frac{\{\lambda(w)\}^k}{\mu'(w)}O\Big(\sum_{q=1}^{\infty}\frac{1}{q}\Big|\binom{k}{q}\Big|\Big)=O\Big(\frac{\{\lambda(w)\}^k}{\mu'(w)}\Big).$$

Thus the integrand of $L_R(w)$ is

$$(3)\qquad\Big|\frac{k}{2\sin(1/2)t\{\lambda(w)\}^k}\Big[\frac{\{\lambda(w)\}^k}{k}\sin wt+\frac{\{\lambda(w)\}^k}{\mu'(w)}\cdot O(t)+O(\{\lambda(w)\}^k\{\sin(3/4)t\})\Big]\Big|$$

$$=\Big|\frac{\sin wt}{2\sin(1/2)t}\Big|+O\Big(\frac{1}{\mu'(w)}\Big)+O(1).$$

We shall now estimate the Lebesgue constant $L_R(w)$, which may be written as follows:

$$\frac{2}{\pi}\Big(\int_0^{\mu'(w)}+\int_{\mu'(w)}^{\pi}\Big)\Big|\frac{k}{2\sin(1/2)t\{\lambda(w)\}^k}\int_0^w\{\lambda(w)-\lambda(x)\}^{k-1}\lambda'(x)\sin([x]+1/2)t\,dx\Big|dt.$$

Using (3) in $\int_0^{\mu'(w)}$ and (1), (2) in $\int_{\mu'(w)}^{\pi}$ respectively, we get, when $k\geqq 1$,

$$(4)\quad L_R(w)=\frac{2}{\pi}\int_0^{\mu'(w)}\left|\frac{\sin wt}{2\sin(1/2)t}+O\Big(\frac{1}{\mu'(w)}\Big)\right|dt$$
$$+\frac{2}{\pi}\int_{\mu'(w)}^{\pi}O\Big(\frac{1}{2\sin(1/2)t}\Big\{\frac{1}{\lambda(w)}+\mu'(w)t^{-1}\Big\}\Big)dt$$
$$=\frac{2}{\pi}\int_0^{\mu'(w)}\left|\frac{\sin wt}{2\sin(1/2)t}\right|dt+\frac{2}{\pi}\int_0^{\mu'(w)}O\Big(\frac{1}{\mu'(w)}\Big)dt$$
$$+\frac{2}{\pi}\int_{\mu'(w)}^{\pi}O\Big(\frac{1}{2\sin(1/2)t}\Big\{\frac{1}{\lambda(w)}+\mu'(w)t^{-1}\Big\}\Big)dt$$
$$=\frac{2}{\pi}\int_0^{\mu'(w)}\left|\frac{\sin wt}{2\sin(1/2)t}\right|dt+O(1)+\frac{1}{\lambda(w)}O(\log\mu'(w))+O(1)$$
$$=\frac{2}{\pi}\int_0^{\mu'(w)}\left|\frac{\sin wt}{2\sin(1/2)t}\right|dt+O(1),$$

where the relation $|\log\mu'(w)|=O(\lambda(w))$[2] is used.
When $1>k>0$, since

$$\frac{2}{\pi}\int_{\mu'(w)}^{\pi}O\Big(\frac{1}{2\sin(1/2)t}\Big[\frac{1}{\lambda(w)}+\{\mu'(w)\}^k t^{-k}\Big]\Big)dt=O(1),$$

we get the same result with (4).

Let us take an integer l such that $\pi l/w<\mu'(w)<\pi(l+1)/w$, then we have

$$L_R(w)=\frac{2}{\pi}\sum_{h=0}^{l-1}\int_{\frac{h\pi}{w}}^{\frac{(h+1)\pi}{w}}\left|\frac{\sin wt}{2\sin(1/2)t}\right|dt+\frac{2}{\pi}\int_{\frac{\pi l}{w}}^{\mu'(w)}\left|\frac{\sin wt}{2\sin(1/2)t}\right|dt+O(1)$$
$$=\frac{2}{\pi}\sum_{h=0}^{l-1}\int_0^{\frac{\pi}{w}}\sin wt\left|\frac{1}{2\sin(1/2)(t+h\pi/w)}\right|dt+O(1)$$
$$=\frac{2}{\pi}\int_0^{\frac{\pi}{w}}\sin wt\Big\{\sum_{h=0}^{l-1}\frac{1}{2\sin(1/2)(t+h\pi/w)}\Big\}dt+O(1).$$

As is well known [2], the inner sum is

$$\cong\frac{w}{\pi}\Big\{\log l+O(1)\Big\},$$

and then

$$L_R(w)\cong\frac{4}{\pi^2}\log\{w\mu'(w)\}.$$

References

[1] F. T. Wang: On Riesz summability of Fourier series, Proc. Lond. Math. Soc., **47** (1942).
[2] A. Zygmund: Trigonometrical series, Warszawa (1936).

2) This follows from (3). For, since $\lambda'(w)=\lambda(w)\mu'(w)$, we have for any $A>0$, $1/\mu'(w)\leqq A\lambda(w)$, and then $|\log\mu'(w)|\leqq\log\lambda(w)+\log A\leqq\lambda(w)$.