170. *Note on Algebras of Strongly Unbounded Representation Type. II*

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1. This paper is a continuation of our previous paper[1] on algebras of strongly unbounded representation type. Let A be an algebra over an algebraically closed field k and $g_A(d)$ be the number of inequivalent indecomposable representations of A of degree d where d is a positive integer. Then if A has indecomposable representations of arbitrary high degrees and $g_A(d)=\infty$ for an infinite number of integers d, A is said to be of *strongly unbounded representation type.*

In his paper [1], James P. Jans proved four sufficient conditions[2] for an algebra to be of strongly unbounded representation type and, in our previous paper [3], we added two conditions to them but now in this paper we shall prove another sufficient condition for an algebra to be of strongly unbounded representation type:

(7) *The graph $G(A_0)$ associated with a two sided ideal $A_0 \subset N$ is*

$$\left\{ P_{j_4}, P_{k_5} \,\&\, P_{j_4}, P_{k_5}, P_{k_5} \,\&\, P_{j_3}, P_{j_3}, P_{k_4} \,\&\, P_{j_3}, \begin{matrix} P_{k_4}, P_{k_4} \,\&\, P_{j_2}, \\ P_{k_3}, P_{k_3} \,\&\, P_{j_2}, \end{matrix} P_{j_2}, P_{k_2} \,\&\, P_{j_2}, P_{k_2}, P_{k_2} \,\&\, P_{j_1}, P_{j_1}, P_{k_1} \,\&\, P_{j_1}, P_{k_1} \right\}.$$ [3]

2. First of all we assume that $N^2=0$[4] and A is a basic algebra. In order to prove that this condition is sufficient for an algebra to be of strongly unbounded representation type, by the same way as [3] we construct the matrix function R_{cs}, where $c \in k$ and s is an integer,

$$R_{cs}(a)=\begin{bmatrix} X_T(a) & 0 \\ Y(a) & X_B(a) \end{bmatrix},$$

as follows:

Let $X_T(a)$ be the direct sum of $I_{2s}*X_{j_4}(a)$, $I_{6s}*X_{j_3}(a)$, $I_{11s}*X_{j_2}(a)$ and $I_{5s}*X_{j_1}(a)$ and let $X_B(a)$ be the direct sum of $I_{4s}*X_{k_5}(a)$, $I_{9s}*X_{k_4}(a)$, $I_{5s}*X_{k_3}(a)$, $I_{8s}*X_{k_2}(a)$ and $I_{2s}*X_{k_1}(a)$ where $X_{j_p}(a)$ and $X_{k_q}(a)$ are obtained by the same way as [1] or [3].

Next we put

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1) T. Yoshii [3].

2) James P. Jans [1] or T. Yoshii [3].

3) From now on we use same notations as [1] or [3].

4) James P. Jans [1] for the case where $N^2 \neq 0$.

$$M_{1c}=\begin{bmatrix} \begin{matrix} 1 & c \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 1 & c \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{matrix} \end{bmatrix}, \qquad M_2=\begin{bmatrix} \begin{matrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \end{matrix} \end{bmatrix},$$

$$M_3=\begin{bmatrix} \begin{matrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \end{matrix} \end{bmatrix}, \quad M_4=\begin{bmatrix} \begin{matrix} 1&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 1&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0 \end{matrix} \end{bmatrix},$$

$$M_5=\begin{bmatrix} \begin{matrix} 0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1 \end{matrix} \end{bmatrix},$$

$$M_6=\begin{bmatrix} \begin{matrix} 1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&1 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&1 \end{matrix} \end{bmatrix},$$

$$M_7=\begin{bmatrix} \begin{matrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\\0&0&0&0&1\end{matrix} & & \\ & \ddots & \\ & & \begin{matrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\\0&0&0&0&1\end{matrix} \end{bmatrix} \text{ and } M_8=\begin{bmatrix} \begin{matrix}1&0&0&0&0\\0&0&0&1&1\end{matrix} & & \\ & \ddots & \\ & & \begin{matrix}1&0&0&0&0\\0&0&0&1&1\end{matrix} \end{bmatrix}.$$

Then let $Y(a)$ have $M_{1c}{}^*Y_{54}(a)$ directly below $I_{2s}{}^*X_{j_4}(a)$ and directly to the left of $I_{4s}{}^*X_{k_5}(a)$, $M_2{}^*Y_{53}(a)$ directly below $I_{6s}{}^*X_{j_3}(a)$ and directly to the left of $I_{4s}{}^*X_{k_5}(a)$, $M_3{}^*Y_{43}(a)$ directly below $I_{6s}{}^*X_{j_3}(a)$ and directly to the left of $I_{9s}{}^*X_{k_4}(a)$, $M_4{}^*Y_{42}(a)$ directly below $I_{11s}{}^*X_{j_2}(a)$ and directly to the left of $I_{9s}{}^*X_{k_4}(a)$, $M_5{}^*Y_{32}(a)$ directly below $I_{11s}{}^*X_{j_2}(a)$ and directly to the left of $I_{5s}{}^*X_{k_3}(a)$, $M_6{}^*Y_{22}(a)$ directly below $I_{11s}{}^*X_{j_2}(a)$ and directly to the left of $I_{8s}{}^*X_{k_2}(a)$, $M_7{}^*Y_{21}(a)$ directly below $I_{5s}{}^*X_{j_1}(a)$ and directly to the left of $I_{8s}{}^*X_{k_2}(a)$, $M_8{}^*Y_{11}(a)$ directly below $I_{5s}{}^*X_{j_1}(a)$ and directly to the left of $I_{2s}{}^*X_{k_1}(a)$. Fill out the rest with zeroes.

Then it is shown by the same way as [1] that R_{cs} is a directly indecomposable representation of A.

Next in order to show that A is of strongly unbounded representation type, we prove that R_{cs} and R_{ds} can not be similar for $c \neq d$.

Now suppose that they were similar. Then there would exist a non-singular matrix P intertwining R_{cs} and R_{ds}. Next let P be divided into submatrices,

$$P=\begin{bmatrix} P_{11} & \cdots & P_{19} \\ \vdots & \ddots & \vdots \\ P_{91} & \cdots & P_{99} \end{bmatrix},$$

corresponding to the divisions of R_{cs}. Then it is clear that $P_{12}\cdots P_{19}$, P_{21}, $P_{23}\cdots P_{29}$, P_{31}, P_{32}, $P_{34}\cdots P_{39}$, $P_{41}\cdots P_{43}$, $P_{45}\cdots P_{49}$, $P_{56}\cdots P_{59}$, P_{67}, P_{68}, P_{69}, P_{78}, P_{79}, P_{89} are zero.

Moreover we have

$$P_{55}=\begin{bmatrix} a_{11} & & & & a_{15} & \cdots \\ & a_{22} & & & & \cdot \\ & & a_{33} & & & \cdot \\ & & & a_{44} & & \cdot \\ a_{51} & & & & a_{55} & \\ & a_{62} & & & & \cdot \\ & & \cdot & & & \cdot \\ & & & \cdot & & \cdot \end{bmatrix}$$

and it is impossible from $PR_{cs}(a_{54})=R_{ds}(a_{54})P$ for $a_{54}\in e_{k_5}Ne_{j_4}$ that P

is non-singular.

Thus we have the following

Theorem. *If the graph $G(A_0)$ associated with $A_0 \subset N$ is*

$$\left\{ P_{j_4}, P_{k_5} \& P_{j_4}, P_{k_5}, P_{k_5} \& P_{j_3}, P_{j_3}, P_{k_4} \& P_{j_3}, \begin{matrix} P_{k_4}, P_{k_4} \& P_{j_2}, \\ P_{k_3}, P_{k_3} \& P_{j_2}, \end{matrix} P_{j_2}, P_{k_2} \& P_{j_2}, P_{k_2}, P_{k_2} \& P_{j_1}, P_{j_1}, P_{k_1} \& P_{j_1}, P_{k_1} \right\},$$

A is of strongly unbounded representation type.

Remark

Prof. R. Brauer and Prof. R. M. Thrall have conjectured that the class of algebras of unbounded representation type is only that of algebras of strongly unbounded representation type,[5] but now it is easily shown from the results of [2], [3], and this paper that if $N^2=0$ and k is algebraically closed this conjecture is true.

References

[1] James P. Jans: On the indecomposable representation of algebras, Dissertation, University of Michigan (1954).

[2] T. Yoshii: On algebras of bounded representation type, Osaka Math. Jour., **8**, No. 1 (1956).

[3] T. Yoshii: Note on algebras of strongly unbounded representation type, Proc. Japan Acad., **32**, No. 6 (1956).

5) James P. Jans [1].