1. Fourier Series. V. A Divergence Theorem

By Shin-ichi IZUMI
Department of Mathematics, Hokkaidô University, Sapporo, Japan

(Comm. by Z. SUETUNA, M.J.A., Jan. 12, 1957)

1. In the Mathematical Reviews [1], the following theorem is reviewed.[1)]

Theorem A. *If $f(x)$ is integrable and $f(x)=0$ in a closed set E in $(-\pi, \pi)$, then the Fourier series of $f(x)$ converges to zero in each density point of E, when*

$$(1) \qquad \sum_{k=1}^{\infty} \omega(\delta_k, f) < \infty,$$

(δ_k) being intervals contiguous to E and $\omega(\delta, f)$ denoting the oscillation of f in the interval δ.

K. Tandori [2] proved that, in the above theorem, the condition (1) can not be omitted; that is, there are a closed set E and a continuous function $f(x)$ such that $f(x)=0$ in E, $x=0$ is the density point of E and the Fourier series of $f(x)$ diverges at $x=0$.

We shall here prove that Theorem A is false,[2)] that is,

Theorem 1. *There are a closed set E of positive measure, with $x=0$ as a density point and an integrable function $f(x)$ such that $f(x)=0$ in E and the Fourier series of $f(x)$ diverges at $x=0$ and that the condition (1) is satisfied.*

But Theorem A holds true when the integrability of $f(x)$ is replaced by its continuity. More generally,

Theorem 2. *If $f(x)$ is an integrable function such that $f(x)=0$ in a closed set E in $(-\pi, \pi)$, and*

$$(2) \qquad \sum_{k=1}^{\infty} \omega(\bar{\delta}_k, f) < \infty,$$

where $\bar{\delta}_k$ denotes the closure of δ_k, a contiguous interval of E. Then the Fourier series of $f(x)$ converges to zero in each density point of E.

2. **Proof of Theorem 1.** Let (n_k) be an increasing sequence of integers such that $n_k > n_{k-1}^2$. Let (δ_j) be a sequence of open intervals such that

$$\delta_j \subset (\pi/(2j+1), \pi/2j) \qquad (j=n_k, n_k+1, \cdots, n_k^2)$$

and the length of δ_j is $O(1/j^3)$. We put $E=(-\pi, \pi) - \vee \delta_j$. We define $f(x)$ such that

1) The author could not refer the original paper.

2) We consider $\omega(\delta_k, f)$ as the oscillation of the open interval δ_k

$$\begin{aligned} f(x) &= 1/\delta_j\, j \log^2 j \qquad \text{in } \delta_j \\ &= 0 \qquad\qquad \text{otherwise in } (-\pi, \pi). \end{aligned}$$

Then

$$\int_0^{\pi} f(x)dx \leqq \sum_{j=2}^{\infty} \frac{1}{j \log^2 j} < \infty,$$

and hence $f(x)$ is integrable. Since $\sum \omega(\delta_k, f)=0$, the condition (1) is satisfied. Evidently $x=0$ is a density point of E, and $f(x)=0$ in E.

Now the nth partial sum of the Fourier series of $f(x)$ at $x=0$ is

$$\begin{aligned} s_n(0, f) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{x} \sin nx\, dx + o(1) \\ &= \frac{1}{\pi} \sum_j \int_{\delta_j} \frac{f(x)}{x} \sin nx\, dx + o(1) \\ &= \frac{1}{\pi} \sum_{n_k \leqq j \leqq n_k^2} \int_{\delta_j} \frac{f(x)}{x} \sin nx\, dx + J + o(1) \\ &= I + J + o(1), \end{aligned}$$

say, then we have

$$\begin{aligned} I &\geqq A\, n_k \sum \int_{\delta_j} f(x)\, dx = A\, n_k \sum_{j=n_k}^{n_k^2} \frac{1}{j \log^2 j} \\ &\geqq A\, n_k \left[\frac{-1}{\log j} \right]_{j=n_k}^{n_k^2} \geqq A\, n_k \frac{1}{\log n_k} \to \infty. \end{aligned}$$

Let $\delta_j = (a_j, b_j)$. If $\delta_j \subset (\pi/2n_{k-1}^2, \pi)$, then

$$\begin{aligned} \left| \int_{\delta_j} \frac{f(x)}{x} \sin n_k\, x\, dx \right| &\leqq \frac{A}{a_j} \frac{1}{\delta_j \log^2 j \cdot j} \frac{1}{n_k} \\ &\leqq A \frac{n_{k-1}^2 \cdot n_{k-1}^6}{n_k n_{k-1} \log^2 n_{k-1}} \leqq \frac{A\, n_{k-1}^7}{n_k \log^2 n_{k-1}}, \end{aligned}$$

and hence

$$\left| \sum_{\delta_j \subset (\pi/2n_{k-1}^2, \pi)} \int_{\delta_j} \frac{f(x)}{x} \sin n_k x\, dx \right| \leqq A \frac{n_{k-1}^9}{n_k \log^2 n_{k-1}},$$

which is $o(1)$, when

$$n_k > n_{k-1}^9.$$

Furthermore

$$\begin{aligned} \left| \sum_{\delta_j \subset (0, \pi/n_{k+1})} \int_{\delta_j} \frac{f(x)}{x} \sin n_k x\, dx \right| &\leqq n_k \sum \int_{\delta_j} \frac{1}{\delta_j\, j \log^2 j} \\ &\leqq n_k \sum_{j \geqq n_{k+1}} \frac{1}{j \log^2 j} \leqq \frac{A\, n_k}{\log n_{k+1}} \end{aligned}$$

which is bounded when

(3) $$n_{k+1} > e^{n_k}.$$

If we take (n_k) such that (3) holds, then the Fourier series of $f(x)$ diverges at $x=0$.

3. Proof of Theorem 2. Proof is almost evident. We can suppose that $x=0$ is a density point of E. Then

$$s_n(0, f)=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(x)}{x}\sin nx\,dx+o(1)$$
$$=\frac{1}{\pi}\sum\int_{\delta_k}\frac{f(x)}{x}\sin nx\,dx+o(1),$$

where (δ_k) are contiguous intervals of E in a neighbourhood $\varDelta$ of $x=0$. Let $\delta_k=(a_k, b_k)$. Then

$$\left|\int_{\delta_k}\frac{f(x)}{x}\sin nx\,dx\right|\leqq\sum\frac{\delta_k}{a_k}\max_{t\in\delta_k}f(t).$$

By the condition (2),

$$A=\sum\max_{t\in\delta_k}|f(t)|<\infty.$$

When $\varDelta$ is taken sufficiently small,

$$b_k-a_k<\varepsilon b_k$$

where ε is sufficiently small. Hence

$$\frac{b_k}{a_k}<\frac{1}{1-\varepsilon},$$

and then

$$\frac{\delta_k}{a_k}=\frac{b_k-a_k}{a_k}=\frac{b_k-a_k}{b_k}\frac{b_k}{a_k}<\frac{\varepsilon}{1-\varepsilon}.$$

Thus

$$\limsup_{n\to\infty}|s_n(0, f)|\leqq\frac{A\varepsilon}{1-\varepsilon}$$

Since ε may be taken as small as we please, we get our theorem.

From above proof, we get the following

Theorem 3. *In Theorem 2, the condition (2) may be replaced by*

$$\sum_{k=1}^{\infty}\frac{1}{\delta_k}\int_{\delta_k}|f(t)|\,dt<\infty.$$

References

[1] A. G. Džwarsejšvilli: Math. Reviews, **14**, 635 (1953); Zentralblatt für Math., **41**, 33 (1952).

[2] K. Tandori: Acta de Szeged, **15** (1954).