21. A Note on Homotopy Classification and Extension

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1. Let Y be a topological space such that

$$\pi_i(Y)=0 \quad \text{for } 0\leq i<n,\ n<i<q,\ q<i<r \quad (n>1).$$

When $r<2q-1$, K. Mizuno has studied the obstruction and the classification theorems for mappings of a geometric complex into Y along the line of Eilenberg-MacLane [1]. Our purpose of this note is to generalize these theorems for the case where $r\geq 2q-1$. This paper makes full use the notations of the paper by K. Mizuno [3].

2. Let Π and Π' be abelian groups. For a given cocycle $k\in Z^{q+1}(\Pi, n; \Pi')$, let $K(\Pi, n, \Pi', q; k)$ be the complex defined in the paper [2]. Let (K, L_i), $i=0, 1,\cdots, r\geq 0$, be c.s.s. pairs. Denote by D the subcomplex of $K(\Pi, n, \Pi', q; k)$ generated by all $(1_{p,n}, 1_{p,q})$. As was shown by K. Mizuno [3], a mapping T: $(K, L_0)\rightarrow(K(\Pi, n, \Pi', q; k), D)$ is characterized by a cocycle and a cochain

$$x_n=T^{\#}b_n\in Z^n(K, L_0; \Pi), \qquad c_q=T^{\#}b_q\in C^q(K, L_0; \Pi')$$

subject to $kT(x_n)+\delta c_q=0$ [3, p. 56]. The map T corresponding in this fashion to the pair (x_n, c_q) will be denoted by $T(x_n, c_q)$. Given r-cocycles $x_{q_i}\in Z^{q_i}(K, L_i; \Pi')$ with $q_i\leq q$, $i=1,\cdots, r$, we shall define a chain mapping

$$R_{n,q}(x_n, c_q; x_{q_1},\cdots, x_{q_r}):(K, L)\rightarrow K(\Pi, n, \Pi', q; k)$$

of degree $s=\sum_{i=1}^{r}(q-q_i)$ which is called the defect, where $L=\bigcup_{i=0}^{r}L_i$. The map $R_{r,q}$ is defined as the composite of the maps displayed in the following main diagram:

$$\begin{array}{l}
(K, L)\\
\quad\downarrow e\\
(K, L_0)\times(\mathop{\times}\limits_{i=1}^{r}(K, L_i)) \xrightarrow{\ J\ } (K, L_0)\otimes(\bigotimes_{i=1}^{r}(K, L_i))
\end{array}$$

$$\begin{array}{r}
(K, L_0)\otimes(\bigotimes_{i=1}^{r}(K, L_i))\\
\downarrow R(x_n, c)\otimes(\bigotimes_{i=1}^{r}R(x_{q_i}))\\
K(\Pi, n, \Pi', q; k)\otimes(\bigotimes_{i=1}^{r}(K(\Pi', q_i)))\\
\downarrow I\otimes(\bigotimes_{i=1}^{r}S^{q-q_i})\\
K(\Pi, n, \Pi', q; k)\otimes(\bigotimes_{i=1}^{r}(K(\Pi', q)))
\end{array}$$

$$\begin{array}{l}
K(\Pi, n, \Pi', q; k)\times(\mathop{\times}\limits_{i=1}^{r}K(\Pi', q)) \xleftarrow{\ g\ } K(\Pi, n, \Pi', q; k)\otimes(\bigotimes_{i=1}^{r}(K(\Pi', q)))\\
\quad\downarrow \Upsilon\\
K(\Pi, n, \Pi', q; k).
\end{array}$$

Here, the first map e is the diagonal map. The second map f is the standard map of the Cartesian product into the tensor product. The third map is the tensor product of the FD-maps $R(X_n, c)=T(x_n, c)-T(0,0)$ and $R(x_{q_i})=T(x_{q_i})-T(0)$, $i=1,\cdots,r$, while the fourth map is the tensor product of the suspensions. The fifth map g is the standard map of the tensor into the Cartesian product. Finally, the map γ is defined by

$$\gamma((\phi, \psi)\times\psi_1\times\cdots\times\psi_r)=(\phi, \psi\circ\psi_1\circ\cdots\circ\psi_r),$$

where

$$\psi\circ\psi_1\circ\cdots\circ\psi_r(\beta)=\psi(\beta)+\psi_1(\beta)+\cdots+\psi_r(\beta),$$

for arbitrary appropriate dimensional map β.

3. Let G be an abelian group and let $y\in H^t(\Pi, n, \Pi', q, k; G)$ be a cohomology class. Let X_{q_i} be the cohomology class of x_{q_i}. We shall define the $\cap$-operation $y_\cap(x_n, c; X_{q_1},\cdots, X_{q_r})$ by

$$y_\cap(x_n, c; X_{q_1},\cdots, X_{q_r})=R_{n,q}(x_n, c; x_{q_1},\cdots, x_{q_r})^*y,$$

where $R_{n,q}(x_n, c; x_{q_1},\cdots, x_{q_r})^*: H^t(\Pi, n, \Pi', q, k; G)\to H^{t-s}(K, L; G)$ is the homomorphism induced by $R_{n,q}(x_n, c; x_{q_1},\cdots, x_{q_r})$.

Lemma 1. Let (K', L_i'), $i=0,1,\cdots,r$, be c.s.s. pairs. If $U_i:(K', L_i')\to(K, L_i)$ are simplicial maps which agree on K' and thus determine a simplicial map $U:(K', L')\to(K, L)$, $L'=\bigcup_{i=0}^{r} L_i'$, then

$$U^*[y_\cap(x_n, c; X_{q_1},\cdots, X_{q_r})]=y_\cap(U_0^\# x_n, U_0^\# c; U_1^* X_{q_1},\cdots, U_r^* X_{q_r}).$$

Lemma 2. Let (K, L, M) be a c.s.s. triple. Given a simplicial map $T(x_n, c):(K, M)\to(K(\Pi, n, \Pi', q; k), D)$, cohomology classes $X_{q_i}\in H^{q_i}(K, M; \Pi')$, $q_i\leq q$, $i=1,\cdots,r$, $X_m\in H^m(L, M; \Pi')$, $m<q$ and $y\in H^t(\Pi, n, \Pi', q, k; G)$, we have

$$\begin{aligned} &y_\cap(x_n, c; X_{q_1},\cdots, X_{q_j}, \delta X_m, X_{q_{j+1}},\cdots, X_{q_r}) \\ &=\wp\Big(\sum_{i>j}(q-q_i)\Big)\delta[y_\cap(i^\# x_n, i^\# c; i^*X_{q_1},\cdots, i^*X_{q_j}, X_m, i^*X_{q_{j+1}},\cdots, i^*X_{q_r})] \\ &\qquad\in H^{t-s}(K, L; G) \end{aligned}$$

where $s=\sum_{i=1}^{r}(q-q_i)+(q-m-1)$, $\wp(a)=(-1)^a$, $\delta X_m\in H^{m+1}(K, L; \Pi')$ and $i:(L, M)\to(K, M)$ is the inclusion map.

4. Let Y be a topological space such that

$$\pi_i(Y)=0 \quad \text{for } i<n,\ n<i<q,\ q<i<r,\ 1<n.$$

For the sake of brevity, we write, in the following, $\pi_n=\pi_n(Y)$, $\pi_q=\pi_q(Y)$ and $\pi_r=\pi_r(Y)$. Let $k_n^{q+1}\in Z^{q+1}(\Pi, n; \Pi')$ be the Eilenberg-MacLane invariant of Y. Then, the operation $y_\cap(x_n, c; X_{q_1},\cdots, X_{q_r})$ is defined by using the complex $K(\pi_n, n, \pi_q, q; k_n^{q+1})$. Let $k_{n,q}^{r+1}\in Z^{r+1}(\pi_n, n, \pi_q, q, k; \pi_r)$ and $k_q^{r+1}\in Z^{r+1}(\pi_q, q; \pi_r)$ be the cocycles defined in § 6 of [3]. Let $\mathfrak{K}_{n,q}^{r+1}$ and $\mathfrak{K}_q^{r+1}$ be the cohomology classes of $k_{n,q}^{r+1}$ and k_q^{r+1}.

Let K be a geometric complex. A map $f: K^n\to Y$ determines a cochain $a_f^n\in C^n(K, \pi_n)$ defined by the standard manner. The cochain a_f^n is a cocycle if and only if the map f admits an extension $f_q: K^q\to Y$

which presents an obstruction cocycle $c^{q+1}(f_q)\in Z^{q+1}(K,\pi_q)$ which is represented by

$$c^{q+1}(f_q)=k_n^{q+1}T(a_f^n)+\delta(l^q f_q).$$

This obstruction $c^{q+1}(f_q)$ is zero if and only if the map f_q admits an extension $f_r:K^r\to Y$ which presents an obstruction cocycle $c^{r+1}(f_r)\in Z^{r+1}(K,\pi_r)$ and

$$c^{r+1}(f_r)=k_{n,q}^{r+1}T(a_f^n, l^q f_q)+\delta(l^r f_r) \qquad [3, \text{ Lemma } 7.1].$$

Let L be a subcomplex of K and let $f:L\to Y$ be a map extendible to a map $f':K^r\cup L\to Y$. The cohomology class $\boldsymbol{Z}^{r+1}(f')$ of the obstruction cocycle $c^{r+1}(f_r)$ depends on the choice of the extension $f'|K^q\cup L$ of f.

Lemma 3. Let $f_1,f_2:K^q\cup L\to Y$ be two extensions of the map f: $K^n\cup L\to Y$, and which are extendible to $K^{q+1}\cup L\to Y$. Then,

$$\boldsymbol{Z}^{r+1}(f_1)-\boldsymbol{Z}^{r+1}(f_2)=\mathfrak{K}_{n,q}^{r+1}{}_\cap(a_f^n, l^q f_2; \boldsymbol{a}^q(f_1,f_2))+\mathfrak{K}_q^{r+1}\vdash\boldsymbol{a}^q(f_1,f_2),$$

where $\boldsymbol{a}^q(f_1,f_2)\in H^q(K,L;\pi_q)$ is the cohomology class of the cocycle $l^q f_1-l^q f_2$.

Theorem 1. Let $f:K^n\cup L\to Y$ and let $g:K^r\cup L\to Y$ be an extension of f. Then, the map f admits an extension $f':K^{r+1}\cup L\to Y$ if and only if there is an element

$$\boldsymbol{e}^q\in H^q(K,L;\pi_q)$$

such that

$$\boldsymbol{Z}^{r+1}(g)+\mathfrak{K}_{n,q}^{r+1}{}_\cap(a_f^n, l^q g; \boldsymbol{e}^q)+\mathfrak{K}_q^{r+1}\vdash\boldsymbol{e}^q=0.$$

Theorem 2. Let L be a subcomplex of K such that dim. $(K-L)\leq r$, let $f_0,f_1:K\to Y$ be two maps which agree on $K^{r-1}\cup L$ and let $d^r(f_0,f_1)$ be their difference cocycle. Then, $f_0\simeq f_1$ rel. L, if and only if there exists a cohomology class

$$\boldsymbol{e}^{q-1}\in H^{q-1}(K,L;\pi_q)$$

such that

$$\boldsymbol{d}^r(f_0,f_1)+\mathfrak{K}_{n,q}^{r+1}{}_\cap(a_{f_0}^n, l^q f_0; \boldsymbol{e}^{q-1})+\mathfrak{K}_q^{r+1}\vdash\boldsymbol{e}^{q-1}=0,$$

where $\boldsymbol{d}^r(f_0,f_1)$ is the cohomology class of $d^r(f_0,f_1)$.

References

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