19. On LC^n Metric Spaces

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1. Introduction. A topological space X is called an LC^n space [7, p. 79] if for any point x of X and any neighborhood U of x there exists a neighborhood V of x such that any continuous mapping $g: s^i \to V$, $i=0,1,\cdots,n$, has an extension $\tilde{g}: E^{i+1} \to U$, where S^i is an i-dimensional sphere and E^{i+1} is an $(i+1)$-dimensional element with the boundary S^i. A topological space X is called a C^n space [7, p. 78] if any continuous mapping $g: s^i \to X$, $i=0,1,\cdots,n$, has an extension $\tilde{g}: E^{i+1} \to X$. A topological space X is called an n-ES (resp. n-NES) [6] for metric spaces if, whenever Y is a metric space, B is a closed subset of Y such that $\dim(Y-B) \leqq n$[1] and g is any continuous mapping B to X, there exists an extension $\tilde{g}$ of g from Y (resp. some neighborhood of B in Y) to X. A metric space X is called an n-AR (resp. n-ANR) for metric spaces if, whenever Y is a metric space in which X is closed and $\dim(Y-X) \leqq n$,[1] X is a retract [1] of Y (resp. some neighborhood of X in Y).

In this paper, we shall prove the following theorems concerning LC^n spaces:

Theorem 1. An n-dimensional metric space[1] is an ANR for metric spaces if and only if it is an LC^n space.

Theorem 2. An n-dimensional LC^n metric space X is an n-ES for metric spaces if and only if $\pi_i(X)=0$, $i=0,1,\cdots,n-1$, and $\pi_n(X)$ is 0 or the weak product of infinite cyclic groups, where $\pi_j(X)$ is the j-dimensional homotopy group of X.

Theorem 3. For a metric space X the following conditions are equivalent:

i) X is an LC^n space.

ii) X is an $(n+1)$-NES for metric spaces.

iii) X is an $(n+1)$-ANR for metric spaces.

Theorem 4. If X is an LC^n metric space, for each integer $i=0, 1,\cdots,n$, the following conditions are equivalent:

i) X is a C^i space.

ii) X is an $(i+1)$-ES for metric spaces.

iii) X is an $(i+1)$-AR for metric spaces.

S. Lefschetz [7] proved Theorem 1 in case X is a compact

1) In this paper, we understand by "dimension" the covering dimension. (For example, see [8, p. 350].)

metric space. C. Kuratowski [6] proved Theorems 3 and 4 in case X is a separable metric space. We shall show that these theorems hold in case X is non-separable, too.

2. Proofs of the theorems. (1) The proof of Theorem 1. Since an ANR for metric spaces is an LC^i space for each integer i, the "only if" part of Theorem 1 is obvious. To prove the "if" part, by [3, (1.4), p. 105], it is sufficient to show that for any positive number ε there exist continuous mappings $\phi: X\to K$, $\psi: K\to X$ such that the mapping $\psi\phi: X\to X$ is ε-homotopic[2)] to the identity mapping, where K is a Whitehead complex [2, p. 516].

We shall say that an open covering $\mathfrak{U}=\{U_\alpha\}$ of X has the LC^n property with respect to an open covering $\mathfrak{V}=\{V_\beta\}$ of X if for each U_α there exists an element V_β such that any continuous mapping $g: s^i\to U_\alpha$, $i=0,1,\cdots,n$, has an extension $\tilde{g}: E^{i+1}\to V_\beta$. For an open covering $\mathfrak{U}=\{U_\alpha\}$ of X, denote by $S\mathfrak{U}$ the open covering $\{St(U_\alpha,\mathfrak{U})\}$, where $St(U_\alpha,\mathfrak{U})=\smile\{U_\gamma \mid U_\gamma\frown U_\alpha\neq\phi,\ U_\gamma\in\mathfrak{U}\}$.

We construct a sequence of open coverings $\{\mathfrak{V}_k, \mathfrak{U}_j^i; k=0,1,2,\cdots, i=1,2,\cdots,n \text{ and } j=n+1,n+2,\cdots\}$ of X such that

1) $S\mathfrak{V}_k$ has the property LC^n with respect to $\mathfrak{V}_{k-1}$, $k=1,2,\cdots,n$;

2) $S\mathfrak{V}_j$ has the property LC^n with respect to $\mathfrak{U}_j^n$, $j=n+1,n+2,\cdots$;

3) $S\mathfrak{U}_{j+1}^1$ has the property LC^n with respect to $\mathfrak{V}_j$, $j=n,n+1,\cdots$;

4) $S\mathfrak{U}_j^i$ has the property LC^n with respect to $\mathfrak{U}_j^{i-1}$, $i=2,3,\cdots,n$ and $j=n+1,n+2,\cdots$;

5) $\mathfrak{V}_k$ is a locally finite covering whose order $\leqq n+1$ and the diameter of each element of $\mathfrak{V}_k<\min(\varepsilon/3,\ 1/2(k+1))$.

Next, we construct the following open covering of the product space Y of X and the open interval $(0,1)$. Take a point (x,t) of Y. Suppose $\frac{1}{i}<t\leqq\frac{1}{i-1}$ or $\frac{1}{i}<1-t\leqq\frac{1}{i-1}$, $i=3,4,\cdots$. If $i\leqq n+1$, we select fixed one element V^{n+1} of $\mathfrak{V}_{n+1}$ containing x. If $i>n+1$, we select fixed one element V^i of $\mathfrak{V}_i$ containing x. Put $\eta_x^t=\rho(x, FV^i)$, where ρ is metric in X and FV means the frontier of V. Denote by $U(x,t)$ the spherical neighborhood of (x,t) in Y with the center (x,t) and the radius η_x^t. Since $\dim Y=n+1$, there exists an open covering $\mathfrak{W}$ of Y such that

1) $\mathfrak{W}$ is a locally finite and star refinement of $\{U(x,t)\mid(x,t)\in Y\}$;

2) the nerve M of $\mathfrak{W}$ is the $(n+1)$-dimensional Whitehead complex.

2) Two continuous mappings $f_0, f_1: X\to Y$ are called ε-homotopic if there exists a continuous mapping H from $X\times I$ to Y such that the diameter of $H(x\times I)\leqq\varepsilon$ for each point x of X and $H|X\times 0=f_0$, $H|X\times 1=f_1$.

We construct Dugundji's space $\Pi = X \times (0 \cup 1) \cup M$ and a continuous mapping $f: X \times I \to \Pi$ [4, (3.1)]. Put $F = X \times \left[\frac{1}{n+1}, \frac{n}{n+1}\right]$, $F_0 = X \times [0, \frac{1}{2}] - \cup \{U(x,t) \mid U(x,t) \cap F \neq \phi\}$ and $F_1 = X \times [\frac{1}{2}, 1] - \cup \{U(x,t) \mid U(x,t) \cap F \neq \phi\}$. Denote by M_i, $i=0,1$, the subcomplex of M spanned by all vertexes $\{w_\alpha\}$ of M such that $f^{-1}(w_\alpha) \cap F_i \neq \phi$. Put $L_i = X \times \{i\} \cup M_i$, $i=0,1$. Denote by L_i^j, $i=0,1$ and $j=0,1,\cdots,n+1$, the set $X \times \{i\} \cup$ the j-section of M_i.

We shall construct a continuous mapping $H_0: L_0 \cup L_1 \to X$ as follows. Put $H_0(x,i)=x$, $i=0,1$. Take a vertex w of L_i, $i=0,1$. Select a fixed point (x,t) of Y such that $f(x,t)=w$. Then we have $t<\frac{1}{n+1}$ or $1-t<\frac{1}{n+1}$. Put $H_0(w)=x$. By [4, (3.1)], H_0 is continuous. Let $\overline{w_0w_1}$ be a 1-simplex of L_i. If we denote element of $\mathfrak{W}$ corresponding to v_j by W_j, $j=0,1$, then $W_0 \cap W_1 \neq \phi$. Therefore, $W_1 \subset St(W_0, \mathfrak{W})$. Since $\mathfrak{W}$ is a star refinement of $\{U(x,t)\}$, there exists $U(x,t)$ containing $St(W_0, \mathfrak{W})$. Let τ be the projection $X \times I \to X$. There exists the largest integer s such that $\tau(U(x,t)) \subset V_\alpha^s$ for $V_\alpha^s \in \mathfrak{V}_s$. Then $s<n+1$. If $f(x_0,t_0)=w_0$ and $f(x_1,t_1)=w_1$, we have $x_0 \cup x_1 \subset V_\alpha^s$. Since $S\mathfrak{V}_s$ has the property LC^n with respect to $\mathfrak{U}_s^n$, we have a continuous mapping μ of $\overline{w_0w_1}$ into an element U_s^n of $\mathfrak{U}_s^n$ such that $\mu \mid w_0 \cup w_1 = H \mid w_0 \cup w_1$. Define H on $\overline{w_0w_1}$ by $H_0(y)=\mu(y)$, $y \in \overline{w_0w_1}$. Thus we have a continuous mapping $H: L_0^1 \cup L_1^1 \to X$. Take a 2-simplex $\overline{w_0w_1w_2}$. By the construction of H_0, there exist $U_{s_1}^n, U_{s_2}^n, U_{s_3}^n$ such that $H(\overline{w_0w_1}) \subset U_{s_1}^n$, $H(\overline{w_1w_2}) \subset U_{s_2}^n$, $H(\overline{w_2w_0}) \subset U_{s_3}^n$. Put $s=\min(s_1,s_2,s_3)$. Since $\bigcap_{i=1}^{3} U_{s_i}^n \neq \phi$ and $S\mathfrak{U}_s^n$ has the property LC^n with respect to $\mathfrak{U}_s^{n-1}$. We have an extension of H_0 from $\overline{w_0w_1w_2}$ into an element U_s^{n-1} of $\mathfrak{U}_s^{n-1}$. Thus we have a continuous mapping $H_0: L_0^2 \cup L_1^2 \to X$. By repeated application of this process, we have $H_0: L_0 \cup L_1 \to X$.

Let K be the nerve of $\mathfrak{V}_{n+1}$ with Whitehead's topology and let ϕ be a canonical mapping of X into K. By a similar way as in the above paragraph, we can construct a continuous mapping ψ of K into X such that for each simplex s of K and for each point x of X there exist elements U and U' of $\mathfrak{V}_n$ such that $\psi(s) \subset U$ and $x \cup \psi\phi(x) \subset U'$. Define $H_1: L_0 \cup L_1 \to X$ by $H_1 \mid L_0 = H_0 \mid L_0$ and $H_1 \mid L_1 = \psi\phi H_0 \mid L_1$. Then there exists an element of $\mathfrak{V}_n$ containing $H_1(s)$ for each s of $L_0 \cup L_1$. Denote by M^j the j-section of M, $j=0,1,\cdots,n+1$. Take a vertex w of $M^0 - \bigcup_{i=0}^{1} L_i$. Select a point x of W, where W is the element of $\mathfrak{W}$ corresponding to w. Define $H_2: L_0 \cup L_1 \cup M^0 \to X$ by putting $H_2 \mid L_0 \cup L_1 = H_1$ and $H_2(w)=x$ for $w \in M^0 - L_0 \cup L_1$. By the construction of coverings $\mathfrak{V}_k$, $k=0,1,\cdots,n$, and the definition of H_2, H_2 is extended to a continuous mapping (we use the same letter H_2) of Π into X such that for each simplex s there exists an element of $\mathfrak{V}_0$ containing $H_2(s)$. Define $H: X\times I\to X$ by $H=H_2 f$. Take a point x of X. Let K_1 be the nerve of the covering $\mathfrak{W}\frown(x\times I)$ which we can consider as a subcomplex of M. Then $f(x\times I)\subset x\times(0\smile 1)\smile K_1$. By the construction of H_2, there exists an element V_0 of $\mathfrak{V}_0$ such that $H_2(K_1)\subset St(V_0,\mathfrak{V}_0)$. But the diameter of $St(V_0,\mathfrak{V}_0)<\frac{\varepsilon}{3}\cdot 3=\varepsilon$. Therefore, the homotopy H is an ε-homotopy. This completes the proof of Theorem 1.

(2) The proof of Theorem 2. The "if" part is a consequence of Theorem 4 which we shall prove in the next section. To prove the "only if" part, by the same way as in the proof of Theorem 1, we can construct an n-dimensional complex P and mappings $\phi:(X,x_0)\to(P,p_0)$, $\psi:(P,p_0)\to(X,x_0)$ such that $\psi\phi\simeq 1$ rel (x_0,x_0) in X, where x_0 is a point of X and p_0 is a vertex of P. Since X is an n-ES, we have $\pi_i(X,x_0)=0$, $i<n$. By the well-known Hurewicz's theorem, we have $\pi_n(X,x_0)\approx H_n(X,x_0)$, where $H_n(X,x_0)$ is the n-dimensional homology group of (X,x_0) with the additive group of integers as coefficients. Since $H_n(X,x_0)$ is a direct factor of $H_n(P,p_0)$ and P is the n-dimensional complex, $H_n(X,x_0)$ is 0 or the weak product of infinite cyclic groups. This completes the proof.

(3) The proofs of Theorems 3 and 4. By a similar way as [5, Théorème 2, p. 266] and [6, Théorème 1, p. 273], Theorem 4 is a consequence of the following proposition:

Proposition 1. If X is a (non-separable) metric space, there exists a metric space Y such that

i) X is a closed subset of Y;

ii) $Y-X$ is an infinite complex with the metric topology [2];

iii) whenever Z is a metric space, A is a closed subset of Z and g is a continuous mapping from A to X, g has an extension $\tilde{g}$ from Z to Y.

Proof. According to [9, p. 186], X can be imbedded in a convex subset S of a normed vector space as a closed subset. For each point s of $S-X$, denote by $S(s)$ the spherical neighborhood of s in S with the center s and the radius $\frac{1}{2}\rho(s,X)$, where ρ is metric in S. There exists an open covering $\mathfrak{U}=\{U_\alpha\mid\alpha\in\Omega\}$ of $S-X$ such that

1) $\mathfrak{U}$ is a locally finite star refinement of the covering $\{S(s)\mid s\in S-X\}$;

2) $\mathfrak{U}$ is irreducible, that is, for each α, there exists a point s_α of U_α which does not belong to U_β for any β of Ω, $\beta\neq\alpha$.

Consider the product space $C=S\times\prod_{\alpha\in\Omega} I_\alpha$, where I_α is the half open interval $[0, \infty)$ with the usual topology. Any point of C is represented by $\{s \mid s \in S; k_\alpha \mid k_\alpha \in I_\alpha, \alpha \in \Omega\}$. We identify S with the subset $\{s; k_\alpha = 0 \mid \alpha \in \Omega\}$ of C. Denote by y_α the point $\{s_\alpha; k_\alpha = \rho(s_\alpha, X)$ and $k_\beta = 0$ for $\beta \neq \alpha, \beta \in \Omega\}$ of C. If $\bigcap_{i=0}^{n} U_{\alpha_i} \neq \phi$, there exists a spherical neighborhood $S(s)$ such that $\bigcup_{i=0}^{n} U_{\alpha_i} \subset S(s)$. Therefore, we can construct a simplex $s(\alpha_0, \cdots, \alpha_n)$ in C with the vertexes $y_{\alpha_0}, \cdots, y_{\alpha_n}$. Denote by M the subset $\smile \{s(\alpha_0, \cdots, \alpha_n) \mid (\alpha_0, \cdots, \alpha_n)$ rangs over all finite combinations of elements of Ω such that $\bigcap_{i=0}^{n} U_{\alpha_i} \neq \phi\}$ of C. Put $Y = X \smile M$. Let y_i, $i=1, 2$, be two points of Y such that $y_i \in s(\alpha_0^i, \cdots, \alpha_{n_i}^i)$. Then y_i, $i=1, 2$, are contained in a metric subset $L = S \times \prod_{i=0}^{1} \prod_{j=0}^{n_i} I_{\alpha_j^i}$ of X with the usual metric of the product space. If we define a metric $\tilde{\rho}(y_1, y_2)$ in Y by a metric between y_1 and y_2 in L, Y is a metric subspace of C. Let ϕ be a canonical mapping of $S - X$ to M. Define a mapping $\tilde{\phi}: S \to Y$ by $\tilde{\phi} \mid S - X = \phi$ and $\tilde{\phi} \mid X =$ the identity mapping. If a sequence $\{s_i, i=1, 2, \cdots\}$ of points of $S - X$ has a limit point x in X and $y_i = \tilde{\phi}(s_i) = \{\tilde{s}_i \mid \tilde{s}_i \in S; k_\alpha(i), \alpha \in \Omega: i=1, 2, \cdots\}$, the sequence $\{\tilde{s}_i\}$ has the limit point x. Let $s(\alpha_0^i, \cdots, \alpha_{n_i}^i)$, $i=1, 2, \cdots$, the simplex of M containing y_i. Then we have $\tilde{\rho}(y_i, \tilde{s}_i) \leqq \max \{k_{\alpha_j^i}, j=0, \cdots, n_i\}$, $i=1, 2, \cdots$. Since $\lim_{i \to \infty} \max \{k_{\alpha_j^i}, j=0, \cdots, n_i\} = 0$, the sequence $\{y_i\}$ has the limit point x. Therefore $\tilde{\phi}$ is a continuous mapping.

Let g be a continuous mapping of a closed subset A of a metric space Z to X. Since S is a convex normed vector space, g is an extension g' from Z to S (cf. [4, (3.1)]). Put $\tilde{g} = \tilde{\phi} g'$. Then $\tilde{g}$ is a required extension of g. This completes the proof.

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