# 14. On $\xi$-Rings

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By a $\xi$-ring[1] we mean a ring in which for every element $x$ there exists an element $f(x)$ such that $x-x^2f(x)$ is central. Of course, every strongly regular ring[2] is a $\xi$-ring. Besides, Herstein [5] treated a special type of $\xi$-rings for which $f(x)$ is a polynomial of $x$ with integral coefficients.

A fundamental property of $\xi$-rings is that *every nilpotent element is central.* This is an immediate consequence of Lemma 1. For any subsets $A, B$ of a ring we denote by $A\circ B$ the two-sided ideal generated by all additive commutators $xy-yx$, $x\in A$, $y\in B$.

*Lemma 1. Let $R$ be a ring. Assume that $a\circ R\subseteq(a^2)\circ R$ for every $a\in R$, where $(a^2)$ is the two-sided ideal generated by $a^2$ Then, every nilpotent element of $R$ is central.*[3]

In fact, let $a^n=0$, $n>1$. We shall show that $a$ is central by the induction for $n$. Since $(a^2)^{n-1}=0$, $a^2$ is central, and so $x^{n-1}=0$ for every $x\in(a^2)$. This shows that $x$ is also central and $(a^2)\circ R=0$. Thus, $a\circ R=0$ completing the proof.

It is well known that if every nilpotent element of a ring is central then so is every idempotent.[4] Thus, we see that *every idempotent in a $\xi$-ring is central.*

Another useful result for $\xi$-rings is the following

*Lemma 2. Let $R$ be a $\xi$-ring and $\mathfrak{r}$ its nonzero right ideal. Then $\mathfrak{r}$ contains a nonzero central element.*

In fact, let $0\neq a\in\mathfrak{r}$ and write $f(a)=f$. Then $a-a^2f$ is central, while if $a-a^2f=0$, then $a(a-afa)=(a-a^2f)a=0$, and so $(a-afa)^2=0$, hence $a-afa$ is central. If $a=afa$, then $af=(af)^2$, so $af$ is central. Since we are now supposing $0\neq a=a^2f$, we have $af\neq 0$.

We remark here an evident fact that *every homomorphic image of a $\xi$-ring is also a $\xi$-ring.*

*Lemma 3. In a $\xi$-ring without zero divisors, $af(a)=f(a)a$ for*

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1) This term is due to Dr. M. P. Drazin. We wish to express our gratitude to him who gave us suggestions.

2) A ring is called to be strongly regular if for every $x$ there is $g(x)$ such that $x=x^2g(x)$.

3) Alex Rosenberg proved our Lemma 1 under the assumption that for any $a, b$ there exists $g(a, b)$ such that $b(a-a^2g(a, b))=(a-a^2g(a, b))b$. See Theorem 2 of Drazin [3].

4) See Lemma 2 of Herstein [4] or Theorem 1 of Drazin [3].

*every* $a$.

In fact, we write $f(a)=f$. Since $a-a^2f$ is central, $a^3f=a^2-a(a-a^2f)=a^2-(a-a^2f)a=a^2fa$, so $a^2(af-fa)=0$ which implies $af=fa$.

Our main purpose is to prove the following

*Theorem 1. Let $R$ be a $\xi$-ring. Then the set $N$ of all nilpotent elements of $R$ forms a two-sided ideal contained in the center of $R$. The residue ring $R-N$ is a subdirect sum of division rings and commutative rings.*

Proof. By Lemma 1, $N$ is an ideal contained in the center. Let $x\notin N$. Then there exists a maximum two-sided ideal $P_x$ which contains $N$ and does not contain any powers of $x$. Clearly $P_x$ is a prime ideal. Moreover, by Lemma 2, $R-P_x$ has no zero divisors. First, assume that $R-P_x$ is subdirectly reducible and is a subdirect sum of $R-Q_\alpha$ satisfying $Q_\alpha\supset P_x$. We denote $x$ modulo $P_x$ by $\bar{x}$. Then the $\alpha$-component $\bar{x}_\alpha$ of $\bar{x}$ is nilpotent for every $\alpha$. Since $R-Q_\alpha$ is also a $\xi$-ring, $\bar{x}_\alpha$ is central and moreover $\bar{x}_\alpha u$ is also central for every $u\in R-Q_\alpha$. Hence $\bar{x}$ and $\bar{x}\bar{y}$ are central for every $\bar{y}\in R-P_x$. Let $\bar{z}\in R-P_x$. Then $\bar{x}\bar{y}\bar{z}=\bar{z}\bar{x}\bar{y}=\bar{x}\bar{z}\bar{y}$. Since $\bar{x}\neq 0$ and $R-P_x$ has no zero divisors, we see that $\bar{y}\bar{z}=\bar{z}\bar{y}$, whence $R-P_x$ is a commutative ring. Next, we assume that $R-P_x$ is subdirectly irreducible and denote its unique minimum two-sided ideal by $\mathfrak{a}$. Let $\mathfrak{a}\ni a\neq 0$. Since $a\mathfrak{a}\neq 0$, $a\mathfrak{a}=\mathfrak{a}$ by Lemma 2. Similarly, we obtain $\mathfrak{a}a=\mathfrak{a}$ by Lemma 3. Thus, $\mathfrak{a}$ is a division ring. It follows from this that $R-P_x$ is semisimple and hence is primitive by its subdirect irreducibility. From a well-known theorem for primitive rings we may conclude that $R-P_x$ is a division ring.[5] Since $\bigcap P_x=N$, this completes the proof.

*Corollary. Let $R$ be a $\xi$-ring. Then $af(a)-f(a)a$, $a-f(a)a^2$ and $a-af(a)a$ are all central for every $a\in R$.*

In fact, when $R$ is a division ring, $af(a)=f(a)a$ by Lemma 3. This holds trivially if $R$ is commutative. Thus Corollary follows immediately from Theorem 1.

In the rest of the paper we shall treat $\xi$-rings under a rather strong assumption. A ring is called an I-ring if every nonnil one-sided ideal contains a nonzero idempotent. An FI-ring is a ring of which every homomorphic image is an I-ring.[6]

*Theorem 2. An FI-ring $R$ is a $\xi$-ring if and only if every nilpotent element of $R$ is central. Then, $R-N$ is strongly regular where $N$ is the ideal consisting of all nilpotent elements.*[7]

In fact, we assume that every nilpotent element of an FI-ring

5) See Theorem 22 of Jacobson [6] and its proof.

6) See Levitzki [7]. Of course, every $\pi$-regular ring is an FI-ring.

7) This is a slight generalization of Theorem 6.1 of Drazin [2] and Theorem 5 of Drazin [3].

$R$ is central. Then the FI-ring $R-N$ has no nilpotent elements. Hence $R-N$ is strongly regular[8] and $R$ is a $\xi$-ring.

McLaughlin and Rosenberg proved among others that if all zero divisors of an I-ring $R$ are central then (i) $R$ is commutative or (ii) $R$ is a division ring or (iii) $R$ is a noncommutative ring satisfying the following conditions: The set of zero divisors coincides with the radical $N \neq 0$ and $R-N$ is a (commutative) field.[9] They showed also examples of rings of type (iii)[10] Relating to their results we shall prove finally the following

*Theorem 3. If an FI-ring $R$ is a $\xi$-ring, then $R$ is a subdirect sum of FI-rings in which every zero divisor is central.*

Proof. Let $R$ be a subdirectly irreducible FI$\xi$-ring. We have only to show that any zero divisor of $R$ is central.[11] Denote the unique minimum two-sided ideal of $R$ by $\mathfrak{a}$ and its left annihilator ideal by $l(\mathfrak{a})$. If $l(\mathfrak{a}) \ni e = e^2 \neq 0$, then $eR \supseteq \mathfrak{a}$ since $e$ is central. Thus, $\mathfrak{a} = e\mathfrak{a} = 0$ which is a contradiction. This implies that $l(\mathfrak{a})$ is nil and hence is contained in the center. If $x$ is a left zero divisor, then $x \in l(\mathfrak{a})$ by Lemma 2, so that $x$ is central. Therefore, every right zero divisor is a left zero divisor and is also central. This completes the proof.

## References


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8) See Theorem 5.5 of Levitzki [7].

9) See Theorem 3 of McLaughlin and Rosenberg [8].

10) In such a ring the commutator ideal is nonzero and contained in the radical. Thus, this gives a counterexample for the so-called Herstein's conjecture that every $\xi$-ring is a subdirect sum of division rings and a commutative ring. See Drazin [2].

11) Every ring is a subdirect sum of subdirectly irreducible rings. See Birkhoff [1]

## Postscript

After this paper was presented, we saw Alex Rosenberg: "On a paper of Drazin" (to appear), in which he proved our Theorem 3 under the $\pi$-regularity assumption. We were informed from his letter that he had made the same remark as ours given in the footnote (10), in his review (to appear) of Drazin [2]. Drazin showed that our Lemma 3 is quite unnecessary to prove Theorem 1 since any ring $R \neq 0$ satisfying $xR = R$ for any $x \neq 0$ must be a division ring.