66. On Weakly Compact Regular Spaces. II

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In their paper [4], S. Mardešić and P. Papić have given interesting characterizations of pseudo-compact spaces,[1] and their results lead us to study *weakly compact spaces*[2] introduced by them. The main object of this paper is to give some characterization of weakly compact regular spaces. By a theorem of S. Mardešić and P. Papić, the results stated below give characterizations of pseudo-compact spaces whenever the spaces considered are completely regular.

A topological space[3] E is said to be *weakly compact* if to every pairwise disjoint infinite family of open sets O_α of E there corresponds a point $x \in E$ such that each neighbourhood V of x meets infinitely many O_α. A family of subsets of a topological space E is said to be *locally finite* if each point of E possesses a neighbourhood which meets at most a finite number of the members of the family.

It is known[4] that a completely regular space is pseudo-compact if and only if every locally finite open covering of it has a finite subcovering, or equivalently, every star finite open covering of it has a finite subcovering. This proposition is justified by the following

THEOREM 1. *The following properties of a regular space E are equivalent*:

(1) *E is weakly compact.*

(2) *Every infinite open covering of E has a proper subfamily whose union is dense in E.*

(3) *Every locally finite family $\{O_\alpha\}$ of open sets of E has a finite subfamily whose union contains every O_α.*

(4) *Every locally finite open covering of E has a finite subcovering.*

(5) *Every locally finite open covering of E has a finite subfamily whose union is dense in E.*

(6) *Every star finite open covering of E has a finite subcovering.*[5]

1) A completely regular space is said to be *pseudo-compact* if every continuous function on the space is bounded.

2) "Espaces faiblement compacts". See S. Mardešić and P. Papić [4].

3) Throughout this paper we assume that every topological space satisfies the axiom T_1.

4) See for example K. Iséki and S. Kasahara [2].

5) A covering $\{O_\alpha\}$ of a space is termed *star finite* if each member of $\{O_\alpha\}$ meets only a finite number of O_α's.

(7) *Every star finite open covering of E has a finite subfamily whose union is dense in E.*

Obviously, the given family or coverings in Theorem 1 may be replaced by countable many.

Proof. It is clear that the implications (3)→(4), (4)→(5), (5)→(7), (5)→(6) and (6)→(7) hold true. First we show that (1) implies (2). Let $\{O_\alpha\}$ be an infinite open covering of E, and let us suppose that, for any index β, the union $\bigcup_{\alpha \neq \beta} O_\alpha$ is not dense in E. Then for each index β, the complement U_β of $\overline{\bigcup_{\alpha \neq \beta} O_\alpha}$ is a non-empty open set, and as can easily be seen, the family $\{U_\alpha\}$ is pairwise disjoint. But each point $x \in E$ is contained in some O_α, and O_α does not meet U_β unless $\alpha = \beta$. Therefore the space E is not weakly compact. In order to prove the implication (2)→(3), let us consider a locally finite family $\{O_\alpha\}$ of open sets of E such that, for any finite set J of indices, $\bigcup_{\alpha \in J} O_\alpha \neq \bigcup_\alpha O_\alpha$. Let x_1 be a point in $\bigcup_\alpha O_\alpha$. Since $\{O_\alpha\}$ is locally finite, the point x_1 belongs to only a finite number of O_α's, say $O_1^1, O_2^1, \cdots, O_{m(1)}^1$. We can find then by the regularity of E an open neighbourhood V_1 of x_1 whose closure is contained in O_1^1. Let x_2 be a point in $\bigcup_\alpha O_\alpha$ not belonging to $\bigcup_{i=1}^{m(1)} O_i^1$. It is not hard to see that we can obtain by induction a sequence of points x_n and a sequence of open neighbourhoods V_n of x_n $(n=1, 2, \cdots)$ such that if we denote by $O_1^n, O_2^n, \cdots, O_{m(n)}^n$ all of the members of $\{O_\alpha\}$ containing x_n, then $\overline{V}_n$ is contained in $O_1^n \frown (\bigcup_{i=1}^{n-1} \overline{V}_i)^c$ and $\bigcup_{i=1}^{n-1} \bigcup_{j=1}^{m(i)} O_j^i \not\ni x_n$, where c is the complement operator. Now, if the space E is covered by the sets V_n $(n=1, 2, \cdots)$, then $\{V_n\}$ is an infinite open covering of E which has no proper subfamily whose union is dense in E. On the other hand, if $\bigcup_{n=1}^{\infty} V_n \neq E$, then because of the regularity of the space E, we can choose, for each $n=1, 2, \cdots$, two open sets U_n and N_n such that $\overline{U}_n \subset N_n$ and $\overline{N}_n \subset V_n$. Since $V_n \subset O_1^n$ and since $\{O_\alpha\}$ is locally finite, it can be shown without difficulty that the set $\bigcup_{n=1}^{\infty} \overline{U}_n$ is closed. It follows that the family $\{N_n\}$ forms with $(\bigcup_{n=1}^{\infty} \overline{U}_n)^c$ an infinite open covering of E, but this covering can not possess any proper subfamily whose union is dense in E, proving the implication (2)→(3). It remains only to prove that (7) implies (1). To prove this, suppose that E is not weakly compact; then there exists a pairwise disjoint sequence $\{O_i\}$ of open sets of E which is locally finite. Since the space E is regular, we can find an open set V_1^0 whose closure is contained in O_1, and moreover, for each $n=2, 3, \cdots, n$ open sets $V_n^0, V_n^1, \cdots, V_n^{n-1}$ can be choosen as follows:

$\overline{V_n^0} \subset O_n$ and $\overline{V_n^i} \subset V_n^{i-1}$ for $i=1,2,\cdots,n-1$. We set $U_1 = O_1 \cup (\bigcup_{n=2}^{\infty}(O_n \cap \overline{V_n^1}^c))$ and $U_m = V_m^{m-2} \cup (\bigcup_{n=m+1}^{\infty}(V_n^{m-2} \cap \overline{V_n^m}^c))$ for $m=2,3,\cdots$. Since the set $\bigcup_{i=1}^{\infty} \overline{V_i^0}$ is closed, the sets $\{U_m\}$ $(m=2,3,\cdots)$ and the complement of $\bigcup_{i=1}^{\infty} \overline{V_i^0}$ form an open covering of E which is star finite in view of the construction. But any finite union of the members of this covering can not be dense in E. This completes the proof of the theorem.

In connection with Theorem 1, we note that a normal space in which every point finite open covering has a finite subfamily whose union is dense in the space is nothing but countably compact.

In the remainder of the present paper, we concern with finitely additive monotone set operators. Some of the characterizations of weakly compact spaces obtained by K. Iséki [1] will follow from a theorem of S. Mardešić and P. Papić [4, Théorème 2 or Théorème 3] and the theorems stated below.

It is well known that a countably compact space is characterized in terms of its countable open coverings or decreasing sequences of closed sets etc. On the other hand, S. Mardešić and P. Papić have given two characterizations of weakly compact regular spaces [4, § 2]. Their characterizations are described also in terms of countable open coverings and decreasing sequences of closed sets, and the proofs of them have been done directly from the definition. But indeed, one of their characterizations is an easy consequence of the other, and the procedure of the proof of this fact is similar with the case of countably compact spaces. Moreover, the proof of one of the assertions which the present author has been stated in [3] may be performed analogously. We will discuss here these facts in more general forms, as it seems that this is not without interest.

Let E be a set, and $\mathfrak{S}$ a family of subsets of E closed under the formation of finite unions and finite intersections: $A \cup B \in \mathfrak{S}$ and $A \cap B \in \mathfrak{S}$ whenever $A \in \mathfrak{S}$ and $B \in \mathfrak{S}$. We denote by $\mathfrak{S}^c$ the family consisting of the complements of $A \in \mathfrak{S}$. ϕ will denote the empty set. An operator σ which assigns to each member A of $\mathfrak{S}$ a subset A^σ of E (i.e., an operator defined on $\mathfrak{S}$) is called *additive* if $(A \cup B)^\sigma \subseteq A^\sigma \cup B^\sigma$ for any $A, B \in \mathfrak{S}$, and *monotone* if $A \subseteq B$ implies $A^\sigma \subseteq B^\sigma$ whenever $A, B \in \mathfrak{S}$. An obvious computation shows that an additive operator defined on the class of all subsets of E is monotone. With these notations, we obtain the following

THEOREM 2. *If σ is an additive monotone operator defined on $\mathfrak{S}$, then the following properties are equivalent*:

(1) *Every countable covering $\{A_i\}$ of E which consists of the*

members of $\mathfrak{S}$ has a finite subfamily $A_{i_1}, A_{i_2}, \cdots, A_{i_n}$ such that $\bigcup_{j=1}^{n} A_{i_j}^{\sigma}=E$.

(2) *Every decreasing sequence $\{B_i\}$ of the members of $\mathfrak{S}^c$ such that $B_i^{c\sigma c} \neq \phi$ has a non-empty intersection.*

(3) *Every sequence $\{B_i\}$ of the members of $\mathfrak{S}^c$ such that the sequence $\{B_i^{c\sigma c}\}$ has the finite intersection property has a non-empty intersection.*

Proof. Since the operator σ is monotone, it is clear that (3) implies (2), and the implication (1)$\rightarrow$(3) can be readily shown by way of complementation. To prove the implication (2)$\rightarrow$(1), let us consider a countable covering $\{A_i\}$, $A_i \in \mathfrak{S}$, of E. For any positive integer n, the set $C_n=\bigcup_{i=1}^{n} A_i$ belongs to $\mathfrak{S}$, and we have $(C_n^c)^{c\sigma c}=C_n^{\sigma c}=(\bigcup_{i=1}^{n} A_i)^{\sigma c} \supseteq (\bigcup_{i=1}^{n} A_i^{\sigma})^c$. It follows therefore that if, for any n, the sets $A_1^{\sigma}, A_2^{\sigma}, \cdots, A_n^{\sigma}$ do not cover E, then the set $(C_n^c)^{c\sigma c}$ is not empty, and so we have $\bigcap_{n=1}^{\infty} C_n^c \neq \phi$ or equivalently $\bigcup_{i=1}^{\infty} A_i \neq E$, which is a contradiction. Hence $\bigcup_{i=1}^{n} A_i^{\sigma}=E$ for some n.

THEOREM 3. *If σ is a monotone operator defined on $\mathfrak{S}$, then the following properties are equivalent:*

(1) *From every sequence $\{B_i\}$ of the members of $\mathfrak{S}^c$ such that $\bigcup_{i=1}^{\infty} B_i^{c\sigma c}=E$, we can extract a finite number of B_i's whose union is E.*

(2) *For every decreasing sequence $\{A_i\}$ of non-empty members of $\mathfrak{S}$, we have $\bigcap_{i=1}^{\infty} A_i^{\sigma} \neq \phi$.*

(3) *For every sequence $\{A_i\}$, $A_i \in \mathfrak{S}$, with the finite intersection property, we have $\bigcap_{i=1}^{\infty} A_i^{\sigma} \neq \phi$.*

Proof. It will suffice to show that (2) implies (1). Let $\{B_i\}$ be a sequence of the members of $\mathfrak{S}^c$ such that $\bigcup_{i=1}^{\infty} B_i^{c\sigma c}=E$. For any positive integer n, the set $C_n=\bigcup_{i=1}^{n} B_i$ is a member of $\mathfrak{S}^c$, and consequently $\{C_n^c\}$ $(n=1, 2, \cdots)$ is a decreasing sequence of the members of $\mathfrak{S}$. Therefore, if $C_n \neq E$ for any n, we have $\bigcap_{n=1}^{\infty} C_n^{c\sigma} \neq \phi$. But this contradicts the assumption that the sets $B_i^{c\sigma c}$ cover E, since $C_n^{c\sigma c}=(\bigcap_{i=1}^{n} B_i^c)^{\sigma c} \supseteq (\bigcap_{i=1}^{n} B_i^{c\sigma})^c=\bigcup_{i=1}^{n} B_i^{c\sigma c}$.

Similarly, we have the following theorems.

THEOREM 4. *If σ is an additive operator defined on $\mathfrak{S}$, then the following properties are equivalent:*

(1) *Every covering $\{A_\alpha\}$ of E which consists of the members of*

$\mathfrak{S}$ *has a finite subfamily* $A_{\alpha_1}, A_{\alpha_2}, \cdots, A_{\alpha_n}$ *such that* $\bigcup_{i=1}^{n} A_{\alpha_i}^{\sigma} = E$.

(2) *Every family* $\{B_\alpha\}$ *of the members of* $\mathfrak{S}^c$ *such that* $\{B_\alpha^{c\sigma c}\}$ *has the finite intersection property has a non-empty intersection.*

THEOREM 5. *If* σ *is an operator defined on* $\mathfrak{S}$, *then the following properties are equivalent*:

(1) *From every family* $\{B_\alpha\}$ *of the members of* $\mathfrak{S}^c$ *such that* $\bigcup_\alpha B_\alpha^{c\sigma c} = E$, *we can extract a finite number of* B_α*'s whose union is* E.

(2) *For every family* $\{A_\alpha\}$ *of the members of* $\mathfrak{S}$ *with the finite intersection property, we have* $\bigcap_\alpha A_\alpha^{\sigma} \neq \phi$.

Let σ be an additive operator defined on the class of all subsets of E. If the operator σ satisfies the condition $A^\sigma \supseteq A$ for any subset A of E, and if we adopt as $\mathfrak{S}^c$ the class of all subsets A of E specified by the relation $A = A^\sigma$, then the properties mentioned in Theorems 2 and 3 are all equivalent. In particular, if E is a regular topological space, by taking as σ the closure operator on E and as $\mathfrak{S}$ the class of all open sets in E, every one of the properties mentioned in Theorems 2 and 3 becomes equivalent to the proposition that the space E is weakly compact, by a theorem of S. Mardešić and P. Papić.

References

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