65. On Weakly Compact Regular Spaces. I

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In their paper [3], S. Mardešić and P. Papić introduced a new kind of space, weakly compact topological space.

Let Φ be a family of subsets in topological space S. A point x of S is called *a point accumulation of* Φ, if every neighbourhood of x meets infinitely many members of Φ.

Definition. A topological space S is called *weakly compact*, if any family of disjoint non-empty open sets has at least one point of accumulation. S. Mardešić and P. Papić proved the following interesting proposition: *A regular space is weakly compact if and only if any open countable covering has the AU property in the sense of present writer.*

The proposition gives some elementary properties of a weakly compact regular space, which are analogous for the case of an absolute closed space (cf. M. Katětov [1]).

Theorem 1. A necessary and sufficient condition that a regular space S be weakly compact is that for every family of countable open sets G_n $(n=1,2,\cdots)$ of S having the finite intersection property the intersection of all $\overline{G}_n$ is non-empty.

Proof. To prove the necessity of Theorem 1, let G_n be a given family of countable open sets having the finite intersection property. Suppose that $\bigcap_{n=1}^{\infty}\overline{G}_n=\phi$, then $S=S-\bigcap_{n=1}^{\infty}\overline{G}_n=\bigcup_{n=1}^{\infty}(S-\overline{G}_n)$ and $S-\overline{G}_n$ $(n=1,2,\cdots)$ is an open covering of S. Hence we can find an index N such that $\overline{\bigcup_{n=1}^{N}S-\overline{G}_n}=S$.

Since each G_n is open, we have $\bigcap_{n=1}^{N}G_n=0$ which is a contradiction.

Conversely, let O_n $(n=1,2,\cdots)$ be disjoint non-empty open sets. Let $G_n=\bigcup_{k=n}^{\infty}O_k$, then $\{G_n\}$ has the finite intersection property. Hence $\bigcap_{n=1}^{\infty}\overline{G}_n\neq\phi$. For p of $\bigcap_{n=1}^{\infty}G_n$, any neighbourhood $V(p)$ of p meets each G_n $(n=1,2,\cdots)$. Therefore $V(p)$ meets infinitely many of O_n. Q.E.D.

Let S be a weakly compact regular space, and suppose that $\{F_n\}$ be a decreasing sequence of closed sets such that $\operatorname{Int}F_n\neq\phi$, where $\operatorname{Int}F$ is the interior of F. Then $\operatorname{Int}F_n$ are non-empty open sets having the finite intersection property, and $F_n\supset\overline{\operatorname{Int}F_n}$ for each n.

By Theorem 1, we have $\bigcap_{n=1}^{\infty} \overline{\operatorname{Int} F_n} \neq \phi$, and therefore $\bigcap_{n=1}^{\infty} F_n \neq \phi$.

By a similar argument, if a regular space S is weakly compact, for every countable family of closed sets $\{F_n\}$ such that the family has the finite intersection property and the interior of each closed set F_n is non-empty, then $\bigcap_{n=1}^{\infty} F_n$ is non-empty.

Conversely, suppose that the conclusion of the statement above for a regular space holds, then we shall consider a countable non-empty open sets G_n having the finite intersection property. From $\operatorname{Int} \overline{G}_n \supset G_n$, $\operatorname{Int} \overline{G}_n$ for all n are non-empty. Hence $\bigcap_{n=1}^{\infty} \overline{G}_n \neq \phi$. By Theorem 1, the space S is weakly compact. Therefore we have

Theorem 2. For a regular space S the following statements are equivalent:

1) *S is weakly compact.*

2) *For every countable non-empty open sets G_n of S having the finite intersection property, the intersection of $\overline{G}_n$ is non-empty.*

3) *For every countable closed sets F_n of S having the finite intersection property, if $\operatorname{Int} F_n \neq \phi$, then $\bigcap_{n=1}^{\infty} F_n \neq \phi$.*

4) *For every decreasing sequence of non-empty open sets G_n, $\bigcap_{=1}^{\infty} \overline{G}_n \neq \phi$.*

5) *For every decreasing sequence of closed sets F_n such that $\operatorname{Int} F_n \neq \phi$, $\bigcap_{n=1}^{\infty} F_n \neq \phi$.*

The characterization 5) was given by S. Mardešić and P. Papić [3, Th. 3].

Theorem 3. If a regular space is a continuous image of a weakly compact regular space, then it is weakly compact.

Proof. Let S_1 be a weakly compact regular space, and let S_2 be the image of S_1 by continuous mapping f. To prove the weakly compactness of S_2, take a countable open covering U_n $(n=1, 2, \cdots)$ of S_2, then $f^{-1}(U_n)$ is a countable open covering of S_1. Since S_1 is regular, there are finite numbers of open sets $f^{-1}(U_{n_i})$ $(i=1, 2, \cdots, k)$ such that $\bigcup_{i=1}^{k} \overline{f^{-1}(U_{n_i})} = S_1$. By the continuity of f, $\bigcup_{i=1}^{k} \overline{U}_{n_i} = S_2$. Hence, S_2 is weakly compact.

Theorem 4. The closure of an open set O in a weakly compact regular space S is weakly compact.

Proof. Let $\{O_n\}$ be a countable open covering of $\overline{O}$, then there are open sets O'_n in S such that $O_n = O'_n \frown \overline{O}$. Therefore O'_n and $S-\overline{O}$ are a countable open covering of S, hence we can find O'_{n_i} $(i=1, 2,$

$\cdots, k)$ such that $\bigcup_{i=1}^{k} \overline{O}'_{n_i} \smile \overline{(S-\overline{O})} = S$. Therefore $\bigcup_{i=1}^{k} \overline{O}'_{n_i} \supset O$, and by $\overline{O \frown \bigcup_{i=1}^{k} O'_{n_i}} = \overline{O \frown \overline{\bigcup_{i=1}^{k} O'_{n_i}}}$, we have

$$\bigcup_{i=1}^{k} \overline{O}_{n_i} \supset \overline{O}.$$

This shows that $\overline{O}$ is weakly compact. As an application of Theorems 3 and 4, we shall show

Theorem 5. Let S_1 be a weakly compact regular space, and let f be a $1-1$ continuous mapping of S_1 onto a regular space S_2. Then O of S_1 is regularly open if and only if $f(O)$ is regular open.

Proof. Let $O_2 = f(O_1)$, then $f(\overline{O}_1)$ is closed by Theorems 3 and 4. Therefore $\overline{O}_2 = f(\overline{O}_1)$ and $\overline{S_2 - \overline{O}_2} = \overline{f(S_1 - \overline{O}_1)}$. Similarly $f(\overline{S_1 - \overline{O}_1})$ is closed and we have $f(\overline{S_1 - \overline{O}_1}) = \overline{S_2 - \overline{O}_2}$. This shows $\text{Int}\, \overline{O}_2 = f\, (\text{Int}\, \overline{O}_1)$.
Q.E.D.

Let R be a dense subset of S. Then R is said to be *paracombinatorially imbedded* in S, if for any finite open sets O_i in R_1, $\bigcap_{i=1}^{n} O_i = \phi$, then $\bigcap_{i=1}^{n} \overline{O}_i \subset R$.

Theorem 6. Let R be paracombinatorially imbedded in a weakly compact regular space S, and let T be a one-to-one continuous image of S by f.

Then $f(R)$ is combinatorially imbedded in T.

Proof. Let H_1, H_2 be relatively open sets in $f(R)$ and $H_1 \frown H_2 = \phi$. We shall show $\overline{H}_1 \frown \overline{H}_2 \subset f(R)$. To prove it, let $f(a) = b \in \overline{H}_1 - f(R)$, then, by Theorem 3, $a \in \overline{f^{-1}(H_1)} - R$. Therefore, by the assumption, $a \in S - \overline{f^{-1}(H_2)}$, $b \in T - f(\overline{f^{-1}(H_2)})$. Since $f(\overline{f^{-1}(H_2)})$ is closed, we have $b \in T - \overline{H}_2$. Hence $\overline{H}_1 \frown \overline{H}_2 \subset f(R)$. Q.E.D.

The methods of the proofs of Theorems 4, 5 and 6 are due to M. Katětov [2].

By a result of S. Mardešić and P. Papić [3, Th. 1] and Theorem 1, we have the following

Theorem 7. A necessary and sufficient condition that a completely regular space S be pseudo-compact is that for every family of countable open sets G_n of S having the finite intersection property the intersection of all $\overline{G}_n$ is non-empty.

References

[1] M. Katětov: Über H-abgeschlossene und bikompakte Räume, Cas. Mat. Fys., **69**, 36–49 (1940).

[2] M. Katětov: On H-closed extensions of topological spaces, Cas. Mat. Fys., **72**, 17–32 (1947).

[3] S. Mardešić et P. Papić: Sur les espaces dont toute transformation réelle continue est bornée, Glasnik Mat.-Fiz. i Astr., **10**, 225–232 (1955).