64. A Note on an Inequality of Levitzki

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1. For any ring S we denote the nilpotency index of S modulo its radical by $j(S)$. In his paper [5], J. Levitzki proved the inequality

(1) $$j(R_1+R_2)\leq j(R_1)+j(R_2)$$

which holds for a pair of right ideals R_1, R_2 in every I-ring A. The purpose of the present note is to show a new proof of (1). From our method the following generalization will be derived. For any module N we denote by $m(N)$ the least upper bound of all integers r such that N contains a submodule which is a direct sum of mutually isomorphic r submodules.

Theorem 1. Let M be an S-right module. Assume that for any nonzero x in M there exists a nonzero right ideal R_x which has zero intersection with the annihilator right ideal of x. Then, $m(N_1+N_2)\leq m(N_1)+m(N_2)$ for any submodules N_1, N_2 of M.

First, we note that the inequality (1) can very easily be proved in case the ring A is an FI-ring. In fact, we may assume that $j(R_1)$, $j(R_2)<\infty$ since if not the inequality is trivial. Let P be any primitive ideal of R_1+R_2 and V an irreducible module over $(R_1+R_2)-P$. Then $(R_i+P)-P$ modulo its radical is a dense ring of linear transformations of $((R_i+P)-P)V$ [1, Theorem 2]. From this, since R_i is now also an FI-ring [4, Theorem 2.1], we see that $(R_i+P)-P$ modulo its radical is a total matrix ring, over a division ring, of degree at most $j(R_i)$ [3, Theorem 5.6]. Hence $\dim V\leq \dim((R_1+P)-P)V+\dim((R_2+P)-P)V\leq j(R_1)+j(R_2)$, and so $j(R_1+R_2)=\max\dim V\leq j(R_1)+j(R_2)$.

In the rest of the paper we shall reduce the inequality (1) for an I-ring to that for an FI-ring. Our main tool is the extended centralizer defined by R. E. Johnson [2] and we need a certain number of lemmas relating to it.

2. Let M be an S-right module. We shall use the following notations: M^*=the set of all submodules N of M having the property that $N\cap N'\neq 0$ for all nonzero submodules N' of M; $\mathfrak{R}(M)$=the set of all semi-endomorphisms defined on a member of M^*; $D(\alpha)$=the definition domain of $\alpha\in\mathfrak{R}(M)$; $\mathfrak{C}(M)$=the extended centralizer of S over M; $\bar{\alpha}$=the element of $\mathfrak{C}(M)$ which is the coset containing $\alpha\in\mathfrak{R}(M)$. For any submodule N of M we denote by ΔN the set of all $\bar{\alpha}\in\mathfrak{C}(M)$ corresponding to α such that $J_\alpha\subseteq D(\alpha)$ and $\alpha J_\alpha\subseteq N$ for some $J_\alpha\in M^*$. Let N^c be a maximal submodule of M disjoint to N. The homo-

morphism $\varepsilon_N: N+N^c\to M$, which is the identity on N and vanishes on N^c, belongs to $\mathfrak{R}(M)$. Evidently, $\bar{\varepsilon}_N$ is an idempotent in $\mathfrak{C}(M)$.

Lemma 1. Let N be a submodule of M. Then, $\varDelta N=\bar{\varepsilon}_N\mathfrak{C}(M)$ and $\mathfrak{C}(N)\simeq \varDelta N\bar{\varepsilon}_N$.

The proof is straightforward, and hence will be omitted.

Lemma 2. $j(\mathfrak{C}(M))\leq m(M)$.

Proof. $\mathfrak{C}(M)$ is a regular ring [2, Theorem 2], and so semisimple. Let $c\in\mathfrak{C}(M)$ be a nilpotent element of index r. Then, for all $1\leq i<r$ there exist $\gamma_i\in\mathfrak{R}(M)$ such that $\bar{\gamma}_i=c$ and $\gamma_iD(\gamma_i)\subseteq D(\gamma_{i+1})$. Clearly, $J=\bigcap D(\gamma_i)\in M^*$. Hence, for all $1\leq i<r$ we may find $\delta_i\in\mathfrak{R}(M)$ such that $\delta_i\leq\gamma_i$ and $\delta_iD(\delta_i)\subseteq D(\delta_{i+1})\subseteq J$. We write $\beta_i=\delta_i\delta_{i-1}\cdots\delta_1$ and $\alpha_{ji}=\gamma_{j+r-i}\gamma_{j+r-i-1}\cdots\gamma_{j+1}\beta_i$ for $0\leq j<i<r$. Then, $\bar{\alpha}_{ji}=c^r=0$. Let K be the intersection of the kernels of α_{ji} for $0\leq j<i<r$. Evidently $K\in M^*$. Since $\bar{\beta}_{r-1}=c^{r-1}\neq 0$, the set I of all $x\in K$ satisfying $\beta_{r-1}x=0$ does not belong to M^*. Hence, $H\cap I=0$ for some nonzero submodule H of K. Now, the sum $H+\beta_1H+\cdots+\beta_{r-1}H$ is direct. In fact, if $\beta_ix_i=\beta_{i+1}x_{i+1}+\cdots+\beta_{r-1}x_{r-1}$, then $\beta_{r-1}x_i=\gamma_{r-1}\cdots\gamma_{i+1}\beta_ix_i=\sum_{i<j}\gamma_{r-1}\cdots\gamma_{r+i-j+1}\alpha_{ij}x_j=0$, since $x_j\in H\subseteq K$. Hence $x_i\in I\cap H=0$. Finally we have to show that $H\simeq\beta_iH$. Let $\beta_ix=0$ for some $x\in H$. Then $\beta_{r-1}x=0$, so $x\in H\cap I=0$.

Theorem 2. Let N be a submodule of M. Then $m(N)=j(\varDelta N)$.

Proof. Let N contain a submodule K which is a direct sum of n copies of a module L. Then, $\mathfrak{C}(K)$ is isomorphic to the total matrix ring of degree n over $\mathfrak{C}(L)$ [6, (8.1)]. Hence $j(\mathfrak{C}(K))\geq n$, and so $j(\mathfrak{C}(N))\geq n$ by Lemma 1, which implies that $j(\mathfrak{C}(N))\geq m(N)$. Thus $j(\mathfrak{C}(N))=m(N)$ by Lemma 2. $j(\mathfrak{C}(N))=j(\varDelta N\bar{\varepsilon}_N)=j(\varDelta N)$. Therefore $m(N)=j(\varDelta N)$, completing the proof.

Lemma 3. Let M be an S-right module satisfying the assumption of Theorem 1. Then $\varDelta(N_1+N_2)=\varDelta N_1+\varDelta N_2$ for any submodules N_1, N_2 of M.

Proof. In case $N_1\cap N_2=0$, it is easy to see that $\varDelta(N_1+N_2)\subseteq\varDelta N_1+\varDelta N_2$. If not, let $a\in N_1$, $b\in N_2$ and $a+b\neq 0$. We denote by N_3 a maximal submodule of N_2 disjoint to $N_1\cap N_2$. Then the set T of all $y\in S$ such that $by\in(N_1\cap N_2)+N_3$ belongs to S^* by the argument similar to that of Johnson [2, the top of p. 892]. Hence $(a+b)T\neq 0$ from the assumption. $0\neq(a+b)z=az+bz\in N_1+(N_1\cap N_2)+N_3=N_1+N_3$. This shows that $(N_1+N_2)^*\ni N_1+N_3$. It follows easily that $\varDelta(N_1+N_2)=\varDelta(N_1+N_3)$. Hence $\varDelta(N_1+N_2)\subseteq\varDelta N_1+\varDelta N_3$. Since the $\varDelta$-operation is evidently monotonous, we have $\varDelta N_3\subseteq\varDelta N_2$ and $\varDelta N_1+\varDelta N_2\subseteq\varDelta(N_1+N_2)$. Therefore $\varDelta N_1+\varDelta N_2=\varDelta(N_1+N_2)$ which completes the proof.

Proof of Theorem 1. $\mathfrak{C}(M)$ is regular, and hence is an FI-ring. Therefore, from the inequality (1) for an FI-ring which we have already proved we see that $j(\varDelta N_1+\varDelta N_2)\leq j(\varDelta N_1)+j(\varDelta N_2)$. Hence

$m(N_1+N_2)=j(\Delta(N_1+N_2))=j(\Delta N_1+\Delta N_2)\leq j(\Delta N_1)+j(\Delta N_2)=m(N_1)+m(N_2)$.

3. Lemma 4. Let S be a semisimple I-ring. Then $j(S)=m(S)$.

Proof. First we note that the singular ideal of any semisimple I-ring is zero [6, (4.10)]. In other words, the S-right module S satisfies the assumption of Theorem 1. Hence, $\mathfrak{S}(S)$ may be regarded as a quotient ring in the sense of [2] and we have $j(S)=j(\mathfrak{S}(S))$ by virtue of [6, Theorem 5]. Now, $j(\mathfrak{S}(S))=m(S)$ from Theorem 2. Therefore $j(S)=m(S)$, completing the proof.

Theorem 3. Let R be a right ideal of a semisimple I-ring S. Then $j(R)=m(R)$.

Proof. It is clear that the radical $N(R)$ of R is the left annihilator of R in R. From this it follows easily that $m(R)\leq m(R-N(R))$. Since $R-N(R)$ is also a semisimple I-ring [4], we have $j(R)=j(R-N(R))=m(R-N(R))$ by Lemma 4. It is immediate from [3, Theorem 2.1] that $j(R)\leq m(R)$. Therefore $j(R)=m(R)$, completing the proof.

In view of this theorem, the inequality (1) for any I-ring A follows readily from Theorem 1.

References

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