82. *The Geometry of Lattices by B-covers*

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We have studied certain properties of B-covers in lattices as a generalization of metric betweeness in a normed lattice [2–5]. In this note we shall consider various geometrical properties in lattices by B-covers, B^*-covers and $B^\dagger$-covers.

§ 1. Preliminaries

We shall use the following definition and lemmas which were obtained in [5].

$B(a, b)=\{x \mid x=(a\frown x)\smile(b\frown x)=(a\smile x)\frown(b\smile x)\ a, b, x\in L\}$ is the B-cover of a and b in a lattice L, and if $c\in B(a, b)$, then we shall write acb.

Lemma 1. axb implies $x\frown(a\smile b)=x=x\smile(a\frown b)$.

Lemma 2. axb implies $a\frown x\geqq a\frown b$, $a\smile b\geqq a\smile x$.

Lemma 3. ax_ib $(i=1, 2)$, ax_1x_2 imply x_1x_2b.

Lemma 4. axb, byc, abc imply xby.

Lemma 5. axb, byc, abc imply $a\frown y\leqq x\frown y$.

Lemma 6. (G) is equivalent to (G*) in a modular lattice,
where (G) $(a\frown c)\smile(b\frown c)=c=(a\smile c)\frown(b\smile c)$,
(G*) $(a\frown c)\smile(b\frown c)=c=c\smile(a\frown b)$.

Lemma 7. If L is modular, then $B(a, b)$ is a sublattice.

Lemma 8. In case L is modular, abc, axb, byc imply axc, ayc.

Lemma 9. In case L is modular, abc, axb, byc imply xyc, axy.

Lemma 10. In order that L be a distributive lattice it is necessary and sufficient that the condition (A) below holds for any elements a, b of L.

(A) $x\in B(a, b)$ if and only if $a\frown b\leqq x\leqq a\smile b$.

Lemma 11. For any elements a, b, c, d of L,

(1) $B(a, b)=B(c, d)$ implies $a\smile b=c\smile d$, $a\frown b=c\frown d$ in any lattice L;

(2) $a\smile b=c\smile d$, $a\frown b=c\frown d$ imply $B(a, b)=B(c, d)$, if and only if L is a distributive lattice.

Lemma 12. In case L is a complemented distributive lattice with $0, I$, then we have $B(a, a')=B(0, I)=L$, where $a\frown a'=0$, $a\smile a'=I$.

§ 2. Relations between some B-covers

(1) *abc implies* $(a\frown b)b(b\frown c)$ *and* $(a\smile b)b(b\smile c)$.

(2) $(a\frown b)b(b\frown c)$ *and* $(a\smile b)b(b\smile c)$ *imply abc.*

(3) *abc implies* $a(a\smile b)c$ *and* $a(a\frown b)c$.

Proof. Since (1), (2) are easy, we shall prove (3). We have $b=$

$(a \frown b) \smile (b \frown c) \leqq a \smile (b \frown c)$ from abc, hence we have $a \smile b \leqq a \smile (b \frown c) \leqq a \smile b$, and hence $a \smile b = a \smile (b \frown c) = a \smile (a \frown c) \smile (b \frown c) \leqq a \smile ((a \smile b) \frown c) \leqq a \smile b \smile ((a \smile b) \frown c) = a \smile b$, and consequently $a \smile ((a \smile b) \frown c) = a \smile b$, that is, $a(a \smile b)c$. The second part of (3) may be proved similarly. It follows that $x \in B(a, b)$ implies $a \smile x$, $a \frown x \in B(a, b)$.

(4) *Let $B(a, b) = X$, $B(b, c) = Y$, $B(c, d) = Z$, and assume that abc, acd, then we have*

① *bcd in any lattice,*

② *abd in a modular lattice,*

③ *XYZ in a modular lattice.*

Proof. ① is implied by Lemma 4, and ② by Lemma 8. To prove ③, we take $x \in X$, $y \in Y$, $z \in Z$; then bcd, byc imply byd by Lemma 8, abd, byd imply ayd by Lemma 8, abc, axb, byc imply axy by Lemma 9, ayd, axy imply xyd by Lemma 4, bcd, byc, czd imply yzd by Lemma 9, and hence xyd, yzd imply xyz by Lemma 4.

(5) *In case L is modular, abo, aod, ocd imply XoY, where $B(a, b) = X$, $B(c, d) = Y$.*

Proof. If we take $x \in X$, $y \in Y$, then ocd, cyd imply oyd by Lemma 8, aod, oyd imply aoy by Lemma 4, abo, axb imply axo by Lemma 8, and consequently axo, aoy imply xoy by Lemma 4.

§ 3. B^*-covers and $B^\dagger$-covers

We shall define the B^*-cover and the $B^\dagger$-cover of a and b in a lattice L as follows; $B^*(a, b) = \{x \mid abx, a, b, x \in L\}$, $B^\dagger(a, b) = \{x \mid bax, a, b, x \in L\}$, $B^\dagger(a, B^*(a, b)) = \{y \mid xay \text{ for all } x \in B^*(a, b)\}$, etc.

(1) $B^\dagger(a, B^*(a, b)) \subset B^\dagger(a, b)$.

(2) $B(a, B(a, b)) \subset B(a, b)$, $B^*(a, B^*(a, b)) \subset B^*(a, b)$.

(3) $B(a, B^\dagger(a, b)) \subset B^\dagger(a, b)$.

Proof. Since (1), (2) are trivial, we have only to prove (3). If we take $y \in B(a, B^\dagger(a, b))$, then bax, ayx imply bay by Lemma 4, and hence y belongs to $B^\dagger(a, b)$.

(4) *For any elements a, b of L, we have the following equality.*

$$B^\dagger(a, B(a, b)) = B^\dagger(a, b).$$

Proof. Since it is obvious that $B^\dagger(a, B(a, b)) \subset B^\dagger(a, b)$, we may prove that $B^\dagger(a, B(a, b)) \supset B^\dagger(a, b)$. If we take x from $B^\dagger(a, b)$, then bax with bya implies xay by Lemma 4, hence x belongs to $B^\dagger(a, B(a, b))$.

(5) *In order that L be a modular lattice it is necessary and sufficient that the equality below holds for any elements a, b of L.*

$$B^*(a, B(a, b)) = B^*(a, b), \quad B(a, B^*(a, b)) = B(a, b).$$

Proof. Suppose that L is a modular lattice; then it is obvious that $B^*(a, B(a, b)) \subset B^*(a, b)$, so we have only to prove that $B^*(a, B(a, b)) \supset B^*(a, b)$. If we take x from $B^*(a, b)$, then for any element y of $B(a, b)$ we have ayx by Lemma 8, and hence x belongs to $B^*(a, B(a, b))$. Similarly we have the other equality. If L is not modular, then there exist five elements a, b, c, x, y such that $c=a\frown y=x\frown y$, $b=a\smile y=x\smile y$, $c<a<x<b$. In this case we have $B^*(a, B(a, b))=\{b\}$, $B^*(a, b)=\{b, y\}$, $B(a, b)=\{a, x, b\}$, $B(a, B^*(a, b))=\{a, b\}$, and hence the equality does not hold.

(6) $B^\dagger(a, B(a, B^*(a, b)))=B^\dagger(a, b)$.

Proof. $B(a, b)\supset B(a, B^*(a, b))\ni b$, and hence we have $B^\dagger(a, b)=B^\dagger(a, B(a, b))\subset B^\dagger(a, B(a, B^*(a, b)))\subset B^\dagger(a, b)$ by (4), § 3, and then we have $B^\dagger(a, B(a, B^*(a, b)))=B^\dagger(a, b)$.

§ 4. Structure of lattices

(1) *In a lattice L, suppose that $B^*(b, B(b, a\smile b))=a\smile b$; then*

① *$a\smile b$ is a maximal element;*

② *if a and b are non-comparable, then there is at least one element x in L such that $b<x<a\smile b$ and x does not belong to $B(a, b)$.*

Proof. ① Suppose that $x\geqq a\smile b$; then for any $y\in B(b, a\smile b)$ we have byx from $b\leqq y\leqq a\smile b\leqq x$, and hence x belongs to $B^*(b, B(b, a\smile b))$. Accordingly we have $x=a\smile b$ by hypothesis. ② If $a\smile b$ covers b, then $B(b, a\smile b)=\{b, a\smile b\}$, $B^*(b, B(b, a\smile b))=\{a\smile b, a,\cdots\}$, and hence the equality does not hold. Moreover, if $x_i\in B(a, b)$ for all x_i such that $b<x_i<a\smile b$, then we have $B(b, a\smile b)=\{b, x_i, a\smile b\}$, $B^*(b, B(b, a\smile b))=\{a\smile b, a,\cdots\}$, and hence the equality does not hold, too. This completes the proof.

(2) Following L. R. Wilcox [1], (a, b) was called a modular pair when $x\leqq b$ implies $(x\smile a)\frown b=x\smile(a\frown b)$ and denoted by $(a, b)M$. We shall now define a relative modular pair $(a, b)M^*$ when $a\frown b\leqq x\leqq b$ implies $(x\smile a)\frown b=x\smile(a\frown b)$.

① *$B(b, a\frown b)\subset B(a, b)$ implies $(a, b)M^*$ and conversely $(a, b)M^*$ implies $B(b, a\frown b)\subset B(a, b)$.*

Proof. For x_1 such that $a\frown b\leqq x_1\leqq b$ and ax_1b we have $x_1=(x_1\smile a)\frown(x_1\smile b)=(x_1\smile a)\frown b$; on the other hand we have $x_1\smile(a\frown b)=x_1$ from $x_1\geqq a\frown b$. Thus we have $(a, b)M^*$. Conversely assume that $(a, b)M^*$; then we have $(a\frown x_1)\smile(b\frown x_1)=(a\frown x_1)\smile x_1=x_1$, $(a\smile x_1)\frown(b\smile x_1)=(a\smile x_1)\frown b=x_1\smile(a\frown b)=x_1$, that is, ax_1b.

② *$(a, b)M^*$ implies $(a, b)M$ and conversely $(a, b)M$ implies $(a, b)M^*$.*

Proof. Let $b''\leqq b$. If we put $b'=b''\smile(a\frown b)$, then $a\frown b\leqq b'\leqq b$. Hence we have $b'=b''\smile(a\frown b)\leqq(b''\smile a)\frown b=(b'\smile a)\frown(b''\smile a)\frown b=(b'\smile(a\frown b))\frown(b''\smile a)=b'\frown(b''\smile a)\leqq b'$ by $(a, b)M^*$, and hence we have $(b''\smile a)\frown b=b''\smile(a\frown b)$ for $b''\leqq b$. Consequently we have $(a, b)M$. It is easy to prove the converse.

③ *A lattice L of finite length is a semi-modular lattice whenever $(a, b)M^*$ is symmetric on a and b.*

(3) *Suppose that $B(a, b)\supset B(a, a\frown b)+B(b, a\smile b)$; then $a/a\frown b$ is isomorphic to $a\smile b/b$.*

Proof. From $B(a, b)\supset B(a, a\frown b)$, we have ax_ib $(i=1, 2,\cdots)$ for any $x_i \in B(a, a\frown b)$. Then $b\smile x_i$ belongs to $B(a, b)$ by (3), § 2, and if $b\smile x_1=b\smile x_2$, then we have $x_1=x_2$ from ax_ib $(i=1,2,\cdots)$. Accordingly $x_1 \neq x_2$ implies $b\smile x_1 \neq b\smile x_2$. In the same way, if $y_1 \neq y_2$ where y_i $(i=1,2) \in B(b, a\smile b)$, then we have $a\frown y_1 \neq a\frown y_2$, and $a\frown y_i \in B(a, b)$. Furthermore this mapping preserves order. But we have $a\frown(b\smile x)$ $=(a\smile x)\frown(b\smile x)=x$ for all $x \in B(a, a\frown b)$ by axb, dually $b\smile(a\frown y)=y$ for all $y \in B(b, a\smile b)$, hence the two correspondences are inverses, thus we conclude the proof.

§ 5. Product of B^*-covers

(1) *In any lattice* L, $B^*(a\smile b, b)\frown B^*(a\frown b, b)=B^*(a, b)$.

Proof. It is obtained from (1), (2), § 2.

(2) *In case* L *is modular, let* $B^*(b, a)=X$, $B(a, b)=Y$, $B^*(a, b)$ $=Z$; *then* XbZ *implies* XYZ.

Proof. Let $x \in X$, $y \in Y$, $z \in Z$, and assume that xbz, then xab, ayb imply xyb by Lemma 8, xyb, xbz imply xyz by Lemma 8, this completes the proof.

(3) *We have*

① $B^*(a\frown b, b)\frown B^*(a\smile b, b)\subset B^*(a, a\smile b)\frown B^*(a, a\frown b)$ *in any lattice*;

② $B^*(a\frown b, b)\frown B^*(a\smile b, b)=B^*(a, a\smile b)\frown B^*(a, a\frown b)$ *in a distributive lattice.*

Proof. abx implies $a(a\smile b)x$, $a(a\frown b)x$ by (3), § 2, and $B^*(a, b)=$ $B^*(a\smile b, b)\frown B^*(a\frown b, b)$ from (1), § 5, and hence we have the proof of ①. In case L is distributive, if we take x from the right hand, then we have $a\smile x\geqq a\smile b\geqq b$, $b\geqq a\frown b\geqq a\frown x$ by Lemma 2, and hence by Lemma 10 we have abx since $a\smile x\geqq b\geqq a\frown x$.

(4) *Let* $P=B^*(a\smile b, a\frown b)$, $Q=B^*(a, a\frown b)\frown B^*(b, a\frown b)$, $R=B^*$ $(a\smile b, a)\frown B^*(a\smile b, b)$; *then we have*

① $Q\supset P$ *in any lattice*;

② $Q\supset R=P$ *in a modular lattice*;

③ $Q=R=P$ *in a distributive lattice.*

Proof. ① Let $x \in P$; then $(a\smile b)(a\frown b)x$, $(a\smile b)a(a\frown b)$ imply $a(a\frown b)x$ by Lemma 4, and $(a\smile b)(a\frown b)x$, $(a\smile b)b(a\frown b)$ imply $b(a\frown b)x$ by Lemma 4, hence we have $P\subset Q$. ② By Lemma 7, $(a\smile b)ax$, $(a\smile b)bx$ imply $(a\smile b)(a\frown b)x$, and by Lemma 4, $(a\smile b)(a\frown b)x$, $(a\smile b)a(a\frown b)$, $(a\smile b)b(a\frown b)$ imply $a(a\frown b)x$, $b(a\frown b)x$, hence we have $R\subset P\subset Q$. On the other hand $(a\smile b)(a\frown b)x$, $(a\smile b)a(a\frown b)$, $(a\smile b)b(a\frown b)$ imply $(a\smile b)ax$, $(a\smile b)bx$ by Lemma 8, that is, $P\subset R$. Accordingly we have $Q\supset R=P$. ③ Let $x \in Q$; then we have $a\frown x\leqq a\frown b\leqq a\smile x$, $b\frown x\leqq a\frown b\leqq b\smile x$ by $a(a\frown b)x$, $b(a\frown b)x$ and Lemma 2, and hence we obtain $x\frown(a\smile b)\leqq a\frown b$ $\leqq x\smile(a\smile b)$, that is, $(a\smile b)(a\frown b)x$ by Lemma 10. Consequently we have $Q\subset P$, and hence we obtain $P=Q=R$ by ②.

(5) *Let* a *and* b *of* L *be non-comparable and let* $X_a=\{\alpha \mid a\smile\alpha$

$=b\smile\alpha,\ a\frown\alpha=b\frown\alpha,\ a, b, \alpha\in L\}$.

① $B^*(a, \alpha)=B^*(a\frown b, \alpha)\frown B^*(a\smile b, \alpha)=B^*(b, \alpha)$.

Proof. It is implied from (1), § 5.

② $B^*(a, \alpha)\subset B^*(a, a\smile b)\frown B^*(b, a\smile b)\frown B^*(a, a\frown b)\frown B^*(b, a\frown b)$.

Proof. By (3), § 2 $a\alpha x$ implies $a(a\smile\alpha)x=a(a\smile b)x$, and $a\alpha x$ implies $a(a\frown\alpha)x=a(a\frown b)x$. Since $a\alpha x$ implies $b\alpha x$ and conversely from ①, $a\alpha x$ implies $b(a\smile b)x$ and $b(a\frown b)x$.

③ *In case L is modular, we have* $\sum B^*(a, \alpha_i)=B^*(a, a\smile b)\frown B^*(b, a\smile b)\frown B^*(a, a\frown b)\frown B^*(b, a\frown b)$, *where* $\alpha_i\in X_\alpha$.

Proof. By ② we may prove that if we take x from the right hand, then x belongs to the left hand. We have $a\frown((a\frown b)\smile x)=a\frown b$, $a\smile((a\smile b)\frown x)=a\smile b$ from $a(a\smile b)x, a(a\frown b)x$.
Now let $\beta_0=(a\frown b)\smile((a\smile b)\frown x)=(a\smile b)\frown((a\frown b)\smile x)$, then we have (i) $(a\frown b)\beta_0(a\smile b)$ since $a\frown b\leqq\beta_0\leqq(a\frown b)\smile x$, $(a\smile b)\frown x\leqq\beta_0\leqq a\smile b$, and (ii) $(a\frown b)\beta_0 x$ since $(a\frown b)\smile(\beta_0\frown x)=\beta_0\frown((a\frown b)\smile x)=\beta_0$, and (iii) $(a\smile b)\beta_0 x$ since $(a\smile b)\frown(\beta_0\smile x)=\beta_0\smile((a\smile b)\frown x)=\beta_0$. Hence we have $a\beta_0 x$ from (ii) and (iii) and (1), § 5, where $\beta_0\in X_\alpha$ by (i). This completes the proof.

(6) ① *In any lattice ab_1b_2 implies $B^*(a, b_2)\subset B^*(b_1, b_2)$.*

Proof. By Lemma 4 ab_1b_2, ab_2x imply b_1b_2x.

② *In case L is modular, ab_1b_2 is equivalent to $B^*(a, b_2)\subset B^*(a, b_1)$.*

Proof. By Lemma 8 ab_1b_2, ab_2x imply ab_1x. Conversely if $B^*(a, b_2)\subset B^*(a, b_1)$, then we have ab_1b_2 since $b_2\in B^*(a, b_1)$.

(7) *Let b_1ab_2, $b_1\smile b_2=c$, $b_1\frown b_2=d$; then we have*

① $B^*(a, b_1)\frown B^*(a, b_2)\subset B^*(a, c)\frown B^*(a, d)$ *in a modular lattice;*

② $B^*(a, b_1)\frown B^*(a, b_2)=B^*(a, c)\frown B^*(a, d)$ *in a distributive lattice.*

Proof. ① Assume that b_1ab_2, ab_2x, ab_1x, then we have $a\frown x\leqq d\frown x$ since $a\frown x\leqq b_1$, $a\frown x\leqq b_2$; on the other hand since $b_1\frown b_2=d\leqq a$ from b_1ab_2 we have $d\frown x\leqq a\frown x$, and hence $a\frown x=d\frown x$. Similarly we have $a\smile x=c\smile x$. Thus we have $a\smile(c\frown x)=(a\smile x)\frown c=c$ by modular law and that is, acx. In the same way we have adx. ② *If acx, adx* in a distributive lattice, then we have $a\frown x\leqq b_1$, $b_2\leqq a\smile x$ by Lemma 2, and we obtain ab_1x, ab_2x by Lemma 10.

References

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