81. On Closed Mappings. II

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(Comm. by K. KUNUGI, M.J.A., June 12, 1957)

1. A topological space is said to be locally peripherally compact or semicompact (=semibicompact) if every point has arbitrarily small open neighbourhoods with compact boundaries. The purpose of this note is to establish the following theorems.

Theorem 1. *Let f be a quasi-compact continuous mapping of a locally peripherally compact Hausdorff space X onto a Hausdorff space Y such that, for each point y of Y, the inverse image $f^{-1}(y)$ is connected and the boundary $\mathfrak{B}f^{-1}(y)$ of $f^{-1}(y)$ is compact. Then f is a closed mapping and Y is locally peripherally compact.*

Theorem 2. *Let f be a closed continuous mapping of a locally peripherally compact Hausdorff space X onto a locally peripherally compact Hausdorff space Y such that $\mathfrak{B}f^{-1}(y)$ is compact for each point y of Y. Then f can be extended to a continuous mapping of $\gamma(X)$ onto $\gamma(Y)$, where $\gamma(X)$ and $\gamma(Y)$ mean the Freudenthal compactifications of X and Y respectively.**)

Our Theorem 1 generalizes a theorem of A. H. Stone [6, Theorem 2] as well as a theorem of S. Hanai [2, Theorem 3].

2. **Proof of Theorem 1.** Let X be a locally peripherally compact Hausdorff space. A finite open covering $\{G_1,\cdots,G_r\}$ of X is called a γ-covering of X if $\mathfrak{B}G_i$ is compact for each i. Let $\{\mathfrak{U}_\lambda \mid \lambda\in\Lambda\}$ be the totality of all the γ-coverings of X. Then the following propositions are proved in our previous paper [3].

(1) For any two γ-coverings $\mathfrak{U}_\lambda$ and $\mathfrak{U}_\mu$ there exists a γ-covering $\mathfrak{U}_\nu$ which is a refinement of $\mathfrak{U}_\lambda$ and $\mathfrak{U}_\mu$.

(2) For any γ-covering $\mathfrak{U}_\lambda$ there exists a γ-covering $\mathfrak{U}_\mu$ which is a star-refinement of $\mathfrak{U}_\lambda$.

(3) For each point x of X, $\{S(x,\mathfrak{U}_\lambda) \mid \lambda\in\Lambda\}$ is a basis of neighbourhoods of x.

Now let f be a quasi-compact continuous mapping of X onto a Hausdorff space Y such that, for each point y of Y, $f^{-1}(y)$ is connected and $\mathfrak{B}f^{-1}(y)$ is compact. Let y_0 be any point of Y and let G be any open set of X containing $f^{-1}(y_0)$. Since $\mathfrak{B}f^{-1}(y_0)$ is compact and X is locally peripherally compact, there exist a finite number of open sets H_i, $i=1,\cdots,m$, of X such that $\mathfrak{B}H_i$ is compact and $H_i\subset G$ for each i, and that $\mathfrak{B}f^{-1}(y_0)\subset\smile\{H_i \mid i=1,\cdots,m\}$. Let $G_0=[\smile\{H_i \mid$

*) As for the Freudenthal compactifications, cf. [3].

$i=1,\cdots,m\}]\cup\operatorname{Int} f^{-1}(y_0)$. Then we have

(4) $$f^{-1}(y_0)\subset G_0\subset G$$

and $\mathfrak{B}G_0$ is compact.

Let $\mathfrak{U}_{\lambda_0}$ be an open covering $\{G_0, X-f^{-1}(y_0)\}$ of X. Then $\mathfrak{U}_{\lambda_0}$ is a γ-covering of X since $\mathfrak{B}G_0$ and $\mathfrak{B}f^{-1}(y_0)$ are compact. Let us put

(5) $$W_\lambda=S(\mathfrak{B}f^{-1}(y_0),\mathfrak{U}_\lambda)\cup\operatorname{Int} f^{-1}(y_0),\qquad \lambda\in\Lambda_0.$$

Here we denote by Λ_0 the set of indices $\lambda\in\Lambda$ such that $\mathfrak{U}_\lambda$ is a refinement of $\mathfrak{U}_{\lambda_0}$. Then we have clearly

(6) $$W_\lambda\subset G_0,\qquad \text{for } \lambda\in\Lambda_0.$$

Let $\{V_\alpha(y_0)\mid\alpha\in\Omega\}$ be a basis of open neighbourhoods of y_0 in Y. We shall prove that, for each $\alpha\in\Omega$, there exists an element λ of Λ_0 such that

(7) $$f(W_\lambda)\subset V_\alpha(y_0).$$

For each point x of $\mathfrak{B}f^{-1}(y_0)$ there exists an element $\mu(x)$ of Λ_0 such that

(8) $$f(S(x,\mathfrak{U}^\Delta_{\mu(x)}))\subset V_\alpha(y_0),$$

where $\mathfrak{B}^\Delta$ denotes a covering $\{S(x,\mathfrak{B})\mid x\in X\}$ for any covering $\mathfrak{B}$ (cf. [7]); the existence of such an index $\mu(x)$ is seen from (2), (3) and the continuity of f. Since $\mathfrak{B}f^{-1}(y_0)$ is compact, there exist a finite number of points x_i, $i=1,\cdots,n$, of $\mathfrak{B}f^{-1}(y_0)$ such that

(9) $$\mathfrak{B}f^{-1}(y_0)\subset\cup\{S(x_i,\mathfrak{U}_{\mu_i})\mid i=1,\cdots,n\},$$

where $\mu_i=\mu(x_i)$, $i=1,\cdots,n$. Let $\mathfrak{U}_\lambda$ be a γ-covering of X which is a refinement of $\mathfrak{U}_{\mu_i}$ for each i. Let x be any point of $S(\mathfrak{B}f^{-1}(y_0),\mathfrak{U}_\lambda)$. Then there exists a point x' of $\mathfrak{B}f^{-1}(y_0)$ such that $x\in S(x',\mathfrak{U}_\lambda)$. From (9) it follows that we have $x'\in S(x_i,\mathfrak{U}_{\mu_i})$ for some i. Hence we have

$$x\in S(x',\mathfrak{U}_\lambda)\subset S(S(x_i,\mathfrak{U}_{\mu_i}),\mathfrak{U}_\lambda)\subset S(S(x_i,\mathfrak{U}_{\mu_i}),\mathfrak{U}_{\mu_i})=S(x_i,\mathfrak{U}^\Delta_{\mu_i}),$$

and from (8) we get $f(x)\in V_\alpha(y_0)$ (it is to be noted that $\mu_i=\mu(x_i)$). Thus the existence of $\lambda\in\Lambda_0$ satisfying the condition (7) is proved.

From (7) it follows immediately that

(10) $$\bigcap_{\lambda\in\Lambda_0}\overline{f(W_\lambda)}\subset\bigcap_{\alpha\in\Omega}\overline{V_\alpha(y_0)}.$$

Since Y is a Hausdorff space and $\{V_\alpha(y_0)\mid\alpha\in\Omega\}$ is a basis of open neighbourhoods of y_0, we have $\bigcap_\alpha\overline{V_\alpha(y_0)}=y_0$ and hence

(11) $$\bigcap_{\lambda\in\Lambda_0}\overline{f(W_\lambda)}=y_0.$$

Now we shall prove that there exists some W_λ, $\lambda\in\Lambda_0$ such that

(12) $$f^{-1}(f(W_\lambda))\subset G_0.$$

To prove this, suppose that there exists no such $\lambda\in\Lambda_0$ satisfying (12). Then for each $\lambda\in\Lambda_0$ there exists an element y_λ of Y such that $y_\lambda\in f(W_\lambda)$, $f^{-1}(y_\lambda)\frown(X-G_0)\neq 0$. Since $f^{-1}(y_\lambda)\frown W_\lambda\neq 0$ and $W_\lambda\subset G_0$ (cf. the relation (6)), we have $f^{-1}(y_\lambda)\frown G_0\neq 0$. Since $f^{-1}(y_\lambda)$ is connected by the assumption, we have $f^{-1}(y_\lambda)\frown\mathfrak{B}G_0\neq 0$. Therefore for each $\lambda\in\Lambda_0$ we have

$$(13) \qquad f^{-1}(\overline{f(W_\lambda)}) \frown \mathfrak{B}G_0 \neq 0.$$

Now the family $\{f^{-1}(\overline{f(W_\lambda)}) \frown \mathfrak{B}G_0 \mid \lambda \in \Lambda_0\}$ has the finite intersection property, since we have $W_\mu \subset \bigcap_{j=1}^{s} W_{\lambda_j}$ if $\mathfrak{U}_\mu$ is a refinement of $\mathfrak{U}_{\lambda_j}$ for each j. By the construction of G_0 $\mathfrak{B}G_0$ is compact. Hence we have

$$(14) \qquad [\bigcap_{\lambda \in \Lambda_0} f^{-1}(\overline{f(W_\lambda)})] \frown \mathfrak{B}G_0 \neq 0.$$

On the other hand, from (11) we obtain

$$\bigcap_{\lambda \in \Lambda_0} f^{-1}(\overline{f(W_\lambda)}) = f^{-1}(\bigcap_{\lambda \in \Lambda_0} \overline{f(W_\lambda)}) = f^{-1}(y_0).$$

Hence we have $f^{-1}(y_0) \frown \mathfrak{B}G_0 \neq 0$ from (14), but this is a contradiction to the relation (4). Thus the existence of $\lambda \in \Lambda_0$ satisfying (12) is proved.

The relation (12) shows that if $f^{-1}(y) \frown W_\lambda \neq 0$ then $f^{-1}(y) \subset G_0$. Hence $\{f^{-1}(y) \mid y \in Y\}$ is an upper semi-continuous decomposition of X. Since f is quasi-compact continuous, f is a closed mapping. This proves the first assertion of Theorem 1.

In [6] A. H. Stone has proved that if f is a closed continuous mapping of a locally peripherally compact Hausdorff space X onto a Hausdorff space Y such that, for each point y of Y, $f^{-1}(y)$ is connected and $\mathfrak{B}f^{-1}(y)$ is compact, then Y is locally peripherally compact. Thus we see that Theorem 1 holds.

3. Proof of Theorem 2. As is proved in [5, Lemma 3], if A is a closed set of Y such that $\mathfrak{B}A$ is compact then $\mathfrak{B}f^{-1}(A)$ is compact. Hence by virture of the proof of [4, Theorem 3] we see that f can be extended to a continuous mapping of $\gamma(X)$ onto $\gamma(Y)$.

4. Remarks. As is observed in Stone [6], the condition that $f^{-1}(y)$ be connected for each point y of Y can not be omitted from Theorem 1 even if X is locally compact. If we omit from Theorem 1 the condition that X be locally peripherally compact, we can not conclude that f is a closed mapping; this is seen from [1, p. 70, Example 2]. Likewise we can not conclude the closedness of f without assuming the condition that $\mathfrak{B}f^{-1}(y)$ is compact for each point y of Y, as is remarked by S. Hanai [2, Example 2].

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