80. *On Weakly Compact and Countably Compact Topological Spaces*

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In a series of our papers in Proc. Japan Acad., vol. 33, nos. 2–6 (1957), the present author and S. Kasahara gave some characterisations of weakly compact and countably compact topological spaces. On the other hand, recently T. Isiwata[1] gave some characterisations of countably compact spaces by using the quasi-uniformly continuity in the sense of R. G. Bartle.[2]

In this paper, we shall give new characterisations of these spaces by the concept of proper subcovering of a given open covering. Following S. Mardešić and P. Papić,[3] a topological space is said to be *weakly compact*, if any family of pairwise disjoint open sets of it has at least one cluster point.

Then we have the following

Theorem 1. For a regular T_1-space S, the following propositions are equivalent:

(1) *S is weakly compact.*

(2) *Every point finite countable infinite open covering α has a proper subcovering β such that the union of the closures of elements of β is S.*

(3) *Every point finite countable infinite open covering α has a proper subcovering β such that the closure of the union of elements of β is S.*

Proof. The implications $(1)\to(2)$, (3) are clear by Theorem of my Note,[4] and $(2)\to(3)$ is trivial. To prove $(3)\to(1)$, suppose that S is not weakly compact, then we can find a pairwise disjoint locally finite countable open sets family U_n $(n=1, 2,\cdots)$. By the regularity of S, for each n, there is an open set W_n such that $\overline{W_n}\subseteq U_n$. Since U_n $(n=1, 2,\cdots)$ is locally finite, $\bigcup_{n=1}^{\infty}\overline{W}_n$ is closed, $\bigcup_{n=1}^{\infty}\overline{W}_n=\overline{\bigcup_{n=1}^{\infty}W_n}$ and $\bigcup_{n=1}^{\infty}\overline{U}_n$

1) Cf. T. Isiwata: Some characterizations of countably compact spaces, Sci. Rep. Tokyo Kyoiku Daigaku, **5**, 185–189 (1956).

2) Cf. R. G. Bartle: On compactness in functional analysis, Trans. Amer. Math. Soc., **79**, 35–57 (1955).

3) See S. Mardešić et P. Papić: Sur les espaces dont toute transformation réelle continue est bornée, Glasnik Mat.-Fiz. i. Astr., **10**, 225–232 (1955).

4) See K. Iséki: On weakly compact topological spaces, Proc. Japan Acad., **33**, 182 (1957).

$=\overline{\bigcup_{n=1}^{\infty} U_n}$. Now, if $S=\bigcup_{n=1}^{\infty} U_n$, it is clear that the covering $\alpha=\{U_1, U_2,\cdots\}$ has no proper subcovering, since α is locally finite. If $S\neq\bigcup_{n=1}^{\infty} U_n$, we shall take an open covering $\alpha=\{S-\bigcup_{n=1}^{\infty} \overline{W}_n, U_1, U_2,\cdots\}$. The covering α is point finite and has no proper subcovering such that the closure of the union of elements of it is S. Q.E.D.

Remark. The "*point finite*" in the conditions (2) and (3) may be replaced by the "*locally finite*". It is contained in the proof of Theorem 1 implicitly.

In our paper,[5] we have given two characterisations of a countably compact regular space. We shall show the following

Theorem 2. The following propositions for a regular T_1-space S are equivalent:

(1) *S is countably compact.*

(2) *Every point finite infinite open covering of S has a proper subcovering.*

(3) *Every point finite countable infinite open covering of S has a proper subcovering.*

Proof. It is clear that the implications (1)$\rightarrow$(2) ((1)$\rightarrow$(3))[6] and (2)$\rightarrow$(3). We shall prove the implication (3)$\rightarrow$(1). Suppose that S is not countably compact. Then there is an infinite sequence $\{x_n\}$ $(n=1, 2,\cdots)$ such that $\{x_n\}$ is an isolated set. Therefore, for each n, we can take a neighbourhood (open set) U_n of x_n such that $\{U_n\}$ $(n=1, 2\cdots)$ is pairwise disjoint, and by the regularity of S, there is an open set V_n such that $x_n\in V_n\Subset U_n$ for each n. If $\bigcup_{n=1}^{\infty} U_n=S$, then the covering $\{U_n\}$ has no proper subcovering. Otherwise, we shall consider the open covering $\alpha=\{S-\bigcup_{n=1}^{\infty}\overline{V}_n, U_1, U_2,\cdots\}$. The covering α is point finite countable infinite open covering and has no proper subcovering. Therefore the proof is complete.

5) Cf. K. Iséki and S. Kasahara: On pseudo-compact and countably compact spaces, Proc. Japan Acad., **33**, 100–102 (1957).

6) See K. Iséki and S. Kasahara: Loc. cit., p. 101.