79. A Characterisation of Pseudo-compact Spaces

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(Comm. by K. KUNUGI, M.J.A., June 12, 1957)

Recently, T. Isiwata [5] has given some characterisations of a countably compact normal space by the concept of quasi-uniform continuity. In this Note, we shall give a characterisation of a pseudo-compact completely regular space by locally uniformly convergence. Some types of the convergences of a sequence of functions have been known (see H. Hahn [1, pp. 211-231]).

Let S be a completely regular T_1-space, and suppose that $f_n(x)$ $(n=1,2,\cdots)$ and $f(x)$ are real valued continuous functions on S. The sequence $f_n(x)$ is said to *converge uniformly at a point* x_0 of S, if, for every $\varepsilon>0$, there are an index N and a neighbourhood U of x_0 such that $|f_n(x)-f(x)|<\varepsilon$ for all $n\geq N$ and $x\in U$. Then we shall prove the following

Theorem 1. Let S be a completely regular T_1-space. Then S is pseudo-compact if and only if any sequence $\{f_n(x)\}$ of continuous functions which converges uniformly to a continuous function $f(x)$ at every point of S converges uniformly in S to $f(x)$.

For the concept of pseudo-compactness, see E. Hewitt [2, p. 67].

Proof. Suppose that S is not pseudo-compact, then there is an unbounded continuous function $f(x)$. For each positive integer n, we shall define $f_n(x)$ as

$$f_n(x) = \mathrm{Min}\,(f(x), n),$$

then it is obvious that $f_n(x)$ is continuous.

For a point x_0 of S, by the continuity of $f(x)$, we can find a neighbourhood U of x_0 and a positive integer N such that $f(x)<N$ for all x of U.

Therefore, by the definition of $f_n(x)$, we have $f_n(x)=f(x)$ for x of U and all $n\geq N$. Hence $f_n(x)$ converges uniformly at the point x_0 to $f(x)$. On the other hand, for every $\varepsilon>0$, we can find a point x_0 such that $|f(x_0)-f_n(x_0)|>\varepsilon$, since $f(x)$ is unbounded. Hence $f_n(x)$ does not converge uniformly to $f(x)$.

To prove the converse, we shall use a theorem of S. Mardešić and P. Papić [6]: *A completely regular T_1-space is pseudo-compact, if and only if every countable open covering has AU-covering* (for the definition of AU-covering, see K. Iséki [3]). Let S be a pseudo-compact, completely regular T_1-space, and suppose that $\{f_n(x)\}$ coverges uniformly at every point to $f(x)$. For a given positive $\varepsilon>0$, and each positive integer N we shall consider the set

O_N=the interior of $\{x \,||f_n(x)-f(x)|<\varepsilon,\ n=N, N+1,\cdots\}$.

Then $\{O_N\}$ $(N=1,2,\cdots)$ is open and for x_0 of S, there are an index N and a neighbourhood U of x_0 such that $|f_n(x)-f(x)|<\varepsilon$ for $n\geq N$ and all x of U. Hence $U\subset O_N$ this means that $\{O_N\}$ $(N=1,2,\cdots)$ is a countable open covering of S, and it is easily seen that $\{O_N\}$ is an increasing sequence. Therefore, by the theorem above, there is an index N_0 such that $\bar{O}_{N_0}=S$. For any $n\geq N_0$, $G_n=\{x\,||f_n(x)-f(x)|<\varepsilon\}$ is open and $O_{N_0}\subset G_n$. Then we have $\bar{G}_n\subset\{x\,||f_n(x)-f(x)|\leq\varepsilon\}$. Hence $\bar{O}_{N_0}\subset\bigcap_{n=N_0}^{\infty}\bar{G}_n=\{x\,||f_n(x)-f(x)|\leq\varepsilon,$ for all $n\geq N_0\}$. Therefore, for all x of S, we have $|f_n(x)-f(x)|\leq\varepsilon$ for $n\geq N_0$, and $\{f_n(x)\}$ converges uniformly to $f(x)$. This completes the proof.

From Theorem 1 and a result of E. Hewitt [2, p. 69], we have the following

Theorem 2. A normal T_1-space is countably compact, if and only if any sequence $\{f_n(x)\}$ of continuous functions which converges uniformly to a continuous function $f(x)$ at each point of S converges uniformly to $f(x)$ on S.

From Theorem 1, we have the following Theorem 3 which is a generalisation of the well-known result as *Dini theorem.*

Theorem 3. A completely regular space S is pseudo-compact, if and only if, for every monotone sequence of continuous functions which converges to any continuous function, the convergence is uniform on S.

Theorem 3 follows from the locally uniformly continuity of any monotone increasing sequence of continuous functions, and Theorem 1.

Next we shall reformulate Theorem 1 by the concept of non-uniformity degree of W. F. Osgood. Let $f(x)$ be the limit of a convergent sequence $f_n(x)$ on a topological space S. By the non-uniformity degree $\chi(x_0)$ for $f_n(x)$ of $x_0\in S$, we shall mean the greatest lower bound of all $\varepsilon=\varepsilon(U,N)$ to be $|f_n(x)-f(x)|<\varepsilon$ for all x of some neighbourhood U of x_0 and all n greater than some N. Then we have

Lemma. A sequence of functions $f_n(x)$ converges uniformly at a point x_0 to $f(x)$ if and only if $\chi(x_0)=0$.

From Lemma and Theorem 1, we have following

Theorem 4. A completely regular space S is pseudo-compact if and only if any convergent sequence of continuous functions which converges to a continuous function with $\chi(x)=0$ for each x of S converges uniformly in S.

In my Note [4], we proved that any pseudo-compact complete uniform space is compact. From this result and Theorem 1, we have the following

Theorem 5. A complete uniform space is compact, if and only *if, every sequence of continuous functions which converges uniformly to a continuous function at every point converges uniformly to* $f(x)$.

Let $\{f_n(x)\}$ be a convergent sequence on S and let $f(x)$ be its limit function. Then $f_n(x)$ is said to *converge simply-uniformly at a point* x_0 to $f(x)$, if, for every positive ε and index N, there are an index n $(\geq N)$ and a neighbourhood U of x_0 such that $|f_n(x)-f(x)|<\varepsilon$ for x of U. Under the same hypotheses on $f_n(x)$, $f(x)$, $f_n(x)$ is said to *converge to* $f(x)$ *quasi-uniformly on* S, if for every $\varepsilon>0$ and N, there is a finite number of indices $n_1, \cdots, n_k \geq N$ such that for each x at least one of the following relations holds:

$$|f_{n_i}(x)-f(x)|<\varepsilon \qquad (i=1, 2, \cdots, k).$$

(For these definitions, see H. Hahn [1, pp. 211-214].) Then we have the following

Theorem 6. If a sequence of continuous functions on a pseudo-compact completely regular space S is simply-uniformly convergent at every point of S, then it converges quasi-uniformly on S.

From the definitions, it is clear that *the converse of Theorem 6 holds.* Therefore, these two concepts on a pseudo-compact completely regular space are same one.

The same idea can be used to prove Theorem 6. To prove it, let $f_n(x)$ be continuous functions which converge simply-uniformly to a continuous function $f(x)$ at every point of S.

For a given positive ε and a given index N, we shall consider the set $O_n=\{x\,|\,|f_n(x)-f(x)|<\varepsilon\}$ for each $n\geq N$. Then O_n are open and by the definition of $f_n(x)$, $\{O_n\}$ $(n=N, N+1, \cdots)$ is an open covering of S. Hence we can find a finite number of open sets $O_{n_1}, \cdots, O_{n_k}$ such that $\bigcup_{i=1}^{k} \overline{O}_{n_i}=S$. Therefore for x of S, at least one of the relations:

$$|f_{n_i}(x)-f(x)|\leq\varepsilon \qquad (i=1, 2, \cdots, k)$$

and $n_i\geq N$. This implies the quasi-uniformity.

Hence by Theorem 2 of T. Isiwata [5, p. 187] we have a characterisation of countably compact normal spaces.

Theorem 7. Let S be a normal space. S is countably compact if and only if every sequence of continuous functions which converges to a continuous function at every point of S is simply-uniformly convergent to the continuous function at each point of S.

References

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