76. On the Norms by Uniformly Finite Modulars

By Tetsuya SHIMOGAKI

Mathematical Institute, Hokkaidô University, Sapporo

(Comm. by K. KUNUGI, M.J.A., June 12, 1957)

Let R be a modulared semi-ordered linear space and m be a modular[1] on R. On R we can define two norms as follows:

$$\|x\| = \inf_{\xi>0} \frac{1+m(\xi x)}{\xi}, \quad |||x||| = \inf_{m(\xi x)\leq 1} \frac{1}{|\xi|} \qquad (x\in R).$$

$\|x\|$ is said to be the first norm by m and $|||x|||$ is said to be the second norm or the modular norm by m. Since we have $|||x|||\leqq\|x\| \leqq 2|||x|||$ for every $x\in R$ (cf. [4]), they are equivalent to each other.

It is well known that if a modular m is finite, i.e. $m(x)<+\infty$ for all $x\in R$, then the modular norm is continuous, and that the converse of this is true when R has no atomic element.

In [1] I. Amemiya showed that if a modular m is monotone complete[2] and the modular norm is continuous, then the norm satisfies the following condition: *for every* $1>\varepsilon>0$ *there exists an integer* n *such that the norm of the sum of* n *mutually orthogonal elements having their norm more than* ε *is always* $\geqq 1$. In this paper we call the norm satisfying the above condition to be *finitely monotone.*

We shall investigate the properties of finitely monotone norm and show the form of the conjugate norm in §1. In §2 we examine the relations between a modular and the modular norm in case it is finitely monotone. In fact, we shall prove that if a modular m is uniformly finite, then the modular norm is finitely monotone. The converse of this is valid, if we suppose that R has no atomic element.

If a modular is defined on a universally continuous semi-ordered linear space, then as showed above, we can define the norms whose convergences are equivalent to the modular convergence.[3] Thus it will be conjectured that if a norm is defined on a universally continuous semi-ordered linear space, then there may be defined a modular whose convergence is equivalent to the norm convergence. In §3 we shall establish a normed semi-ordered linear space which is a sort of Köthe space on $[0, 1]$, and it answers negatively to this conjecture. Finitely

1) For the definition of the modular see H. Nakano [4]. The notations and terminologies used here are due to the book [4].

2) m is said to be monotone complete if $0\leqq a_\lambda \uparrow_{\lambda\in\Lambda}$, $\sup_{\lambda\in\Lambda} m(a_\lambda)<+\infty$ implies the existence of $\bigcup_{\lambda\in\Lambda} a_\lambda$.

3) A sequence of elements $x_i\in R$ $(i=1,2,\cdots)$ is said to be modular convergent to x_0, if $\lim_{i\to\infty} m(\xi(x_i-x_0))=0$ for every $\xi>0$.

monotone norm plays essential rôle in constructing this example.

§1. Let R be a universally continuous semi-ordered linear space with a norm $\|x\|$. For convenience, we use a notation "$\oplus\sum_{i=1}^{n} x_i$" in stead of $\sum_{i=1}^{n} x_i$ when $|x_i|\frown|x_j|=0$ for $i\neq j$.

Definition 1.1. *A norm on R is said to be finitely monotone, if for every $\varepsilon>0$, there exists an integer $n_0=n_0(\varepsilon)$ such that $x=\oplus\sum_{i=1}^{n} x_i$, $\|x\|\leqq 1$, $\|x_i\|\geqq\varepsilon$ $(i=1,2,\cdots,n)$ implies $n\leqq n_0$.*

About this definition we can see without difficulty that the words "*for every $\varepsilon>0$*" may be replaced by "*for some $1>\varepsilon>0$*" without changing the meaning of the definition. And we can see also that all norms on finite dimensional spaces are finitely monotone. A norm on R is said to be uniformly monotone, if for any $\gamma,\varepsilon>0$ there exists $\delta>0$ such that $a\frown b=0$, $\|a\|\leqq\gamma$, $\|b\|\geqq\varepsilon$ implies $\|a+b\|\geqq\|a\|+\delta$ (cf. [4, §30]). Now we have

Theorem 1.1. *If a norm on R is uniformly monotone, then it is finitely monotone.*

Proof. For every $\varepsilon>0$, there exists $\delta'>0$ such that $\|x\|\leqq 1$, $\|y\|\geqq\varepsilon$, $x\frown y=0$ implies $\|x+y\|\geqq\|x\|+\delta'$, since $\|x\|$ is uniformly monotone. Then let n_0 be an integer such that $n_0\geqq\frac{1}{\delta'}+1$. If $x=\oplus\sum_{i=1}^{n} x_i$, $\|x\|\leqq 1$, $\|x_i\|\geqq\varepsilon$ $(i=1,2,\cdots,n)$, we have

$$\|x\|\geqq\|\oplus\sum_{i=1}^{n-1} x_i\|+\delta'\geqq\cdots\geqq(n-1)\delta'.$$

Therefore we obtain $n\leqq\frac{1}{\delta'}+1\leqq n_0$, which completes the proof.

The converse of this theorem is not true. The example is easily obtained. The theorems in [3] concerning uniformly monotone norms remain valid, if we replace the assumption of "*uniformly monotone*" by "*finitely monotone*" without adding any difficulty in the proofs.

Truly we can state (cf. [3, Theorems 14.4, 14.7])

Theorem 1.2. *If a norm on R is semi-continuous and finitely monotone, then it is continuous.*

Theorem 1.3. *If a norm on R is finitely monotone and complete, then it is monotone complete and continuous.*

Now we define

Definition 1.2. *A norm on R is said to be finitely flat, if for every $\gamma>0$ there exists $\varepsilon=\varepsilon(\gamma)$ such that $x=\oplus\sum_{i=1}^{n} x_i$, $\|x\|\geqq 1$, $\|x_i\|\leqq\varepsilon$ $(i=1,2,\cdots,n)$ implies $n\geqq\frac{\gamma}{\varepsilon}\|x\|$.*

It is easily seen that all norms on finitely dimensional spaces are always finitely flat. Because, if we choose ε such that $\varepsilon<\frac{1}{N}$, in case of N-dimensional space, we have $\|x\|<1$ for such x that $x=\oplus\sum_{i=1}^{N}x_i$, $\|x_i\|\leqq\varepsilon$ $(i=1,2,\cdots,N)$.

$\overline{R}^{\|}$ denotes the norm conjugate space of R, i.e. the space of norm bounded universally continuous linear functionals on R, and $\|\overline{x}\|(\overline{x}\in\overline{R}^{\|})$ denotes the conjugate norm of $\|x\|$ on R. Then we have

Theorem 1.4. *If a norm on R is finitely monotone, then the conjugate norm is finitely flat.*

Proof. For an arbitrary $\gamma>0$, we choose ε' such that $0<\varepsilon'\gamma<\frac{1}{2}$. For such ε' there exists also an integer $n_0=n_0(\varepsilon')$ (which appears in Definition 1.1), since the norm on R is finitely monotone by assumption. Here we set $\varepsilon=\varepsilon(\gamma)=\frac{1}{6n_0}$. If there exist $\overline{x}_i\in\overline{R}^{\|}$ $(i=0,1,\cdots,n)$ such that $\overline{x}_0=\oplus\sum_{i=1}^{n}\overline{x}_i$, $\|\overline{x}_0\|\geqq1$, $\|\overline{x}_i\|\leqq\varepsilon$ $(i=1,2,\cdots,n)$, then we can find $y\in R$ such that $\|\overline{x}_0\|-\frac{1}{3}<\overline{x}_0(y)$, $\|y\|\leqq1$. Putting $y_i=[\overline{x}_i]^R R$, we have

$$\|\overline{x}_0\|-\frac{1}{3}<\overline{x}_0(y)=\sum_{i=1}^{n}\overline{x}_i(y_i)\leqq\sum_{i=1}^{n}\|\overline{x}_i\|\,\|y_i\|$$

and

$$\frac{2}{3}<n\varepsilon=\frac{n}{6n_0}.$$

This yields $n_0<n$. Since $\|y\|\leqq1$ and $|y_i|\frown|y_j|=0$ for $i\neq j$, there exist y_{i_j} $(j=1,2,\cdots,n-n_0)$ such that $\|y_{i_j}\|<\varepsilon'$ $(j=1,2,\cdots,n-n_0)$. It follows from above that

$$\begin{aligned}\|\overline{x}_0\|-\frac{1}{3}&\leqq\sum_{i=1}^{n}\|\overline{x}_i\|\,\|y_i\|\leqq(n-n_0)\varepsilon\varepsilon'+n_0\varepsilon\\&\leqq n\varepsilon\varepsilon'+\frac{1}{6}\leqq n\varepsilon\varepsilon'+\frac{\|\overline{x}_0\|}{2}-\frac{1}{3}\end{aligned}$$

Thus we obtain $\frac{1}{2}\|\overline{x}_0\|\leqq n\varepsilon\varepsilon'<\frac{n\varepsilon}{2\gamma}$ and $n>\frac{\gamma}{\varepsilon}\|\overline{x}_0\|$. Therefore the conjugate norm is finitely flat by definition.

Theorem 1.5. *If a norm on R is finitely flat, then the conjugate norm is finitely monotone.*

Proof. Let ε be an arbitrary number such that $1>\varepsilon>0$. Now we choose γ' such that $\gamma'\varepsilon\geqq2$. There exists $\varepsilon'=\varepsilon'(\gamma')$ (which appears in Definition 1.2), since $\|x\|$ is finitely flat by assumption. If $\|\overline{x}\|\leqq1$, $\overline{x}=\oplus\sum_{i=1}^{n}\overline{x}_i$, $\|\overline{x}_i\|\geqq\varepsilon$ $(i=1,2,\cdots,n)$, then we can find $0\leqq y_i\in[\overline{x}_i]^R R$ such that $|\overline{x}_i|(y_i)>\frac{\varepsilon\varepsilon'}{2}$, $\|y_i\|\leqq\varepsilon'$. This implies $|\overline{x}|(y)=\sum_{i=1}^{n}|\overline{x}_i|(y_i)>\frac{\varepsilon\varepsilon' n}{2}$, where $y=\oplus\sum_{i=1}^{n}y_i$. $\|y\|\geqq1$ implies $\|\overline{x}\|\,\|y\|>\frac{\varepsilon\varepsilon' n}{2}\geqq\frac{\varepsilon\gamma'}{2}\|y\|\geqq\|y\|$ and $\|\overline{x}\|>1$, which contradicts the assumption on $\overline{x}$. Therefore we obtain

$\|y\|<1$ and $1\geqq\frac{n\varepsilon\varepsilon'}{2}$, which shows $n\leqq\frac{2}{\varepsilon\varepsilon'}$. The proof is completed.

Since the "*conjugate*" of "*uniformly monotone*" is "*uniformly flat*", i.e. for any $\gamma, \varepsilon>0$ there exists $\delta>0$ such that $|a|\frown|b|=0$, $\|a\|\geqq\gamma$, $\|b\|\leqq\delta$ implies $\|a+b\|\leqq\|a\|+\varepsilon\|b\|$ (cf. [4, §31]), we have immediately by the above theorems

Theorem 1.6. *If a norm on R is uniformly flat, then it is finitely flat.*

§2. In this section let a modular m be defined on R and $|||x|||$ $(x\in R)$ be the modular norm by m. Monotone completeness of m will not be assumed here. A modular m is said to be uniformly finite if $\sup\limits_{m(x)\leqq 1} m(\xi x)<\infty$ for every $\xi>0$. Then we have

Theorem 2.1. *If a modular m on R is uniformly finite, then the modular norm is finitely monotone.*

Proof. If $|||x_0|||\leqq 1$, $x_0=\oplus\sum\limits_{i=1}^{n}x_i$, $|||x_i|||\geqq\varepsilon>0$ $(i=1,2,\cdots,n)$, then $|||\frac{1}{\varepsilon}x_0|||\leqq\frac{1}{\varepsilon}$, $|||\frac{1}{\varepsilon}x_i|||\geqq 1$ $(i=1,2,\cdots,n)$. Since $m(a\oplus b)=m(a)+m(b)$ and $|||x|||\geqq 1$ implies $m(x)\geqq 1$, it follows $m\left(\frac{1}{\varepsilon}x_0\right)=\sum\limits_{i=1}^{n}m\left(\frac{1}{\varepsilon}x_i\right)\geqq n$. As m is uniformly finite, we obtain

$$\sup_{m(x)\leqq 1} m\left(\frac{1}{\varepsilon}x\right)=\sup_{|||x|||\leqq 1} m\left(\frac{1}{\varepsilon}x\right)=K_\varepsilon<\infty,$$

which yields $K_\varepsilon\geqq n$. Thus the modular norm is finitely monotone.

Suppose that R has no atomic element. Since $m(x)\geqq N$ implies a partition such that $x=\oplus\sum\limits_{i=1}^{N}x_i$, $m(x_i)\geqq 1$ $(i=1,2,\cdots,N)$ in this case, we obtain obviously

Theorem 2.2. *Suppose that R has no atomic element. If the modular norm by m is finitely monotone, then m is uniformly finite.*

The "*conjugate*" of "*uniformly finite*" is "*uniformly increasing*", i.e. $\lim\limits_{\xi\to\infty}\inf\limits_{m(x)\geqq 1}\frac{m(\xi x)}{\xi}=\infty$ (cf. [4, §48]). Therefore we have by Theorems 1.4 and 2.1

Theorem 2.3. *If a modular m is uniformly increasing, then the modular norm is finitely flat.*

The property called "*finitely monotone*" (*finitely flat*) is a topological one, that is, we have to note

Remark. *If a norm is finitely monotone (finitely flat), then the every norm which is equivalent to it is also finitely monotone (finitely flat).*

This fact can be verified easily by the definitions.

Now we can state

Theorem 2.4. *Suppose that R has no atomic element. If the modular norm by a modular m is finitely flat, then m is uniformly increasing.*

Proof. Since the first norm by m and the modular norm are mutually equivalent, it follows from the above remark that the former is also finitely flat. Since the modular norm by the conjugate modular $\overline{m}$ of m is the conjugate norm of the first norm by m [4, §40], the modular norm $|||\overline{x}|||$ by $\overline{m}$ is finitely monotone by Theorem 1.5, and the conjugate modular $\overline{m}$ is uniformly finite by Theorem 2.2. As the "*conjugate*" of "*uniformly finite*" is "*uniformly increasing*", we obtain Theorem 2.4.

§3. First we shall prove

Theorem 3.1. *Let R be a non-atomic universally continuous semi-ordered linear space, with a norm satisfying the following conditions:*

1) $\|x\|$ *is continuous;*
2) $\|x\|$ *is not finitely monotone;*
3) $\|x\|$ *is monotone complete.*[4]

Then there can not be defined modular on R in any way, whose convergence coincides with that of $\|x\|$.

Proof. Suppose that such a modular m^* is defined on R, then we can define the modular norm $|||x|||^*$ by m^*. It is obvious that the modular m^* is also monotone complete, and $|||x|||^*$ and $\|x\|$ are mutually equivalent (cf. [4, §30]). This implies that $|||x|||^*$ is continuous and the modular m^* is finite, since R has no atomic element. Finiteness and monotone completeness of m^* yield that m^* is uniformly finite (cf. [1]). Then we see that $|||x|||^*$ is finitely monotone, as showed above. This implies that $\|x\|$ is also finitely monotone, which yields a contradiction. Thus the proof is established.

Now we shall show that there exists truly the space which satisfies the conditions of Theorem 3.1. For this purpose we construct a Köthe space X_C on $[0,1]$ (cf. [2]) in the following.

In the sequel e denotes a measurable set in $[0,1]$ and $d(e)$ does the Lebesgue measure of e. We split $[0,1]$ into $e_{i,j}$ ($i=1,2,\cdots;\ j=1,2,\cdots,2^i$) such that

$$\begin{array}{ll} e_1 = e_{1,1} \smile e_{1,2}; & e_i \frown e_j = \phi \ (i \neq j), \\ e_2 = e_{2,1} \smile e_{2,2} \smile e_{2,3} \smile e_{2,4}; & e_{i,k} \frown e_{i,j} = \phi \ (k \neq j), \\ \vdots & d(e_i) = \dfrac{1}{2^i},\ d(e_{i,j}) = \dfrac{1}{2^{2i}} \\ e_n = e_{n,1} \smile e_{n,2} \smile \cdots \smile e_{n,2^n}; & \\ \vdots & (i=1,2,\cdots;\ j=1,2,\cdots,\ 2^i). \end{array}$$

For convenience, $e_{n,0}$ denotes null set ϕ for every $n \geqq 1$. We let λ denote a sequence of measurable sets: $\lambda = \{e_{n,j_n(\lambda)}\}$ $(0 \leqq j_n(\lambda) \leqq 2^n)$, and define that $\lambda = \lambda'$ means $j_n(\lambda) = j_n(\lambda')$ for every $n \geqq 1$. Let Λ be the set of such λ.

4) A norm on R is said to be monotone complete if $0 \leqq a_\lambda \uparrow_{\lambda \in \Lambda}$, $\sup_{\lambda \in \Lambda} \|a_\lambda\| < \infty$ implies the existence of $\bigcup_{\lambda \in \Lambda} a_\lambda$.

Now for every $\lambda \in \Lambda$, we define a measurable function $c_\lambda(t)$ as follows:

$$c_\lambda(t)=\begin{cases} 2^n, \text{ if } t \in e_{n,j_n(\lambda)} & \textit{such that } j_n(\lambda) \neq 0; \\ 0\ , \text{ if } t \in e_n - e_{n,j_n} & \textit{such that } j_n(\lambda) \neq 0; \\ 1\ , \text{ if } t \in e_n & \textit{such that } j_n(\lambda) = 0. \end{cases}$$

Let C be a least convex semi-normal order-closed set including $c_\lambda(t)$ $(\lambda \in \Lambda)$. Then we can see that C satisfies the conditions of Köthe space that has been given in [2], that is,

1) if $c \in C$ and $0 \leqq c_1(t) \leqq c(t)$, then $c_1(t) \in C$, that is, C is semi-normal;
2) if $c_i \in C$, $0 \leqq \alpha_i \leqq 1$, $\sum_{i=1}^{n} \alpha_i = 1$, then $\sum_{i=1}^{n} \alpha_i c_i(t) \in C$;
3) if $c_i \in C$ and $c_i(t) \uparrow c(t)$, then $c \in C$;
4) $1 \in C$, where $1(t)=1$ a.e. on $[0, 1]$;
5) $\int_0^1 c(t)dt \leqq 1$, $c \in C$.

Here we consider the Köthe space $X_C[0, 1]$, i.e. the set of the real valued measurable functions $x(t)$ for which $\| x(t)_X \| = \sup_{c \in C} \int_0^1 | x(t) | c(t)dt < \infty$.

By the definitions of $\| x(t) \|_X$ and of C, we can see that

$$\| x(t) \|_X = \sup_{\lambda \in \Lambda} \int_0^1 | x(t) | c_\lambda(t)dt.$$

It may be easily seen that this Köthe space is a non-atomic universally continuous semi-ordered space and the norm $\| x(t) \|_X$ is monotone complete. In order to see that $\| x(t) \|_X$ is continuous, it is sufficient to prove the following fact: *for any* $x(t) \in X_C\ [0, 1]$ *and* $\varepsilon > 0$ *there exists an integer* n_0 *such that* $\| x(t) \cdot \chi_{[0,1]-\bigcup_{i=1}^{n_0} e_i}(t) \| \leqq \varepsilon$ ($\chi_e(t)$ *means the characteristic function of* e). This fact can be ascertained without difficulty. On the other hand, $\| x(t) \|_X$ is not a finitely monotone norm. For instance, put

$$x_n(t)=\begin{cases} 2^n & \text{on} \quad e_n \\ 0 & \text{otherwise} \end{cases} \qquad (n=1, 2, \cdots).$$

Then we obtain $\| x_n(t) \|_X = 1$ $(n=1, 2, \cdots)$. But $x_n(t)$ can be represented as the forms: $x_n = \oplus \sum_{\nu=1}^{2^n} x_{n,\nu}$, where $x_{n,\nu}(t) = \chi_{e_{n,\nu}}(t)$ and $\| x_{n,\nu}(t) \|_X = 1$ $(n=1, 2, \cdots)$.

Thus this is the space which satisfies the conditions of Theorem 3.1. By virtue of Theorem 3.1 we see that we can define no monotone complete modular on this space.

References

[1] I. Amemiya: A genelalization of the theorem of Orlicz and Birnbaum, Fac. Sci. Hokkaido Univ. Ser. 1, **12**, no. 2, 60–64 (1956).

[2] G. G. Lorentz and D. G. Wertheim: Representation of linear functionals on Köthe spaces, Canad. Jour. Math., **5**, 568–575 (1953).

[3] H. Nakano: Stetige lineare Funktionale auf dem teilweisegeordneten Modul, Jour. Fac. Sci. Imp. Univ. Tokyo, **4**, 201–382 (1942).

[4] H. Nakano: Modulared Semi-ordered Linear Spaces, Tokyo (1950).