96. Fourier Series. XVIII. On a Sequence of Fourier Coefficients

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(Comm. by Z. SUETUNA, M.J.A., July 12, 1957)

1. Let $f(t)$ be an integrable function with period 2π and its Fourier series be

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}A_n(x).$$

Then the derived series is

$$\sum_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx)=\sum_{n=1}^{\infty}B_n(x).$$

L. Fejér [1] (cf. [6, p. 62]) proved that, if $l=f(x+0)-f(x-0)$ exists and is finite, then the sequence $\{nB_n(x)\}$ converges (C,r) $(r>1)$ to l/π. Later many writers treated the Cesàro convergence of the sequence $\{nB_n(x)\}$. Recently B. Singh [2] has proved the following theorem.*)

Theorem. *If*

$$\int_0^t \psi_x(u)\,du=o(t),\qquad \psi_x(t)=f(x+t)-f(x-t)-l,$$

and

$$\lim_{\varepsilon\downarrow 0}\int_\varepsilon^\delta \frac{|\psi_x(t+\varepsilon)-\psi_x(t)|}{t}\,dt=0,$$

where δ is a fixed positive number, then the sequence $\{nB_n(x)\}$ converges $(C,1)$ to the value l/π.

We shall prove the following theorems.

Theorem 1. *Let $0\leqq\alpha\leqq 1$. If*

$$\Psi_x(t)=\int_0^t \psi_x(u)\,du=o\left(t\left(\log\frac{1}{t}\right)^\alpha\right)$$

and

$$\int_0^t(\psi_x(\xi+u)-\psi_x(\xi-u))\,du=o\left(t\Big/\left(\log\frac{1}{t}\right)^{1-\alpha}\right)$$

uniformly in ξ, then $\sigma_n(x)-l/\pi=o((\log n)^\alpha)$ where $\sigma_n(x)$ is the nth $(C,1)$ mean of $\{nB_n(x)\}$.

Theorem 2. *Let $0\leqq\alpha\leqq 1$. If*

$$\Psi_x(t)=o\left(t\left(\log\log\frac{1}{t}\right)^\alpha\right)$$

and

$$\int_0^t(\psi_x(\xi+u)-\psi_x(\xi-u))du=o\left(t\left(\log\log\frac{1}{t}\right)^\alpha\Big/\log\frac{1}{t}\right)$$

uniformly in ξ, then $\sigma_n(x)-l/\pi=o((\log\log n)^\alpha)$.

*) Concerning the earlier references, see [2-4].

Theorem 3. *Let $0 \leqq \alpha \leqq 1$ and $0<r<1$. If*

$$\varPsi_x(t)=o\left(t\left(\log \frac{1}{t}\right)^{\alpha}\right)$$

and
$$\int_0^t (\psi_x(\xi+u)-\psi_x(\xi-u))\, du=o\left(t^{2-r}\left(\log \frac{1}{t}\right)^{\alpha}\right)$$

uniformly in ξ, then $\sigma_n^r(x)-l/\pi=o((\log n)^{\alpha})$, where $\sigma_n^r(x)$ is the nth (C, r) mean of the sequence $\{nB_n(x)\}$.

Theorem 4. *Let $0 \leqq \alpha \leqq 1$ and $0<r<1$. If*

$$\varPsi_x(t)=o\left(t\left(\log\log \frac{1}{t}\right)^{\alpha}\right)$$

and
$$\int_0^t (\psi_x(\xi+u)-\psi_x(\xi-u))\, du=o\left(t^{2-r}\left(\log\log \frac{1}{t}\right)^{\alpha}\right)$$

uniformly in ξ, then $\sigma_n^r(x)-l/\pi=o((\log\log n)^{\alpha})$.

The method of proof is similar to that in our paper [5]. We shall prove Theorems 1 and 3. Proof of the others is similar to above two.

2. Proof of Theorem 1. It is sufficient to prove that

$$\frac{1}{n}\sum_{\nu=1}^{n} \nu B_\nu(x)-\frac{l}{\pi}=\frac{1}{\pi}\int_0^{\pi}\{f(x+t)-f(x-t)-l\}\, g(n, t)\, dt+o(1)$$
$$=\frac{1}{\pi}\int_0^{\pi}\psi_x(t) g(n, t)\, dt+o(1)=o((\log n)^{\alpha}),$$

where
$$g(n, t)=\frac{1}{n}\sum_{\nu=1}^{n}\nu \sin \nu t=-\frac{1}{n}\frac{d}{dt}\left(\frac{1}{2}+\sum_{\nu=1}^{n}\cos \nu t\right)$$
$$=-\frac{1}{n}\frac{d}{dt}\left(\frac{\sin(n+1/2)t}{2\sin t/2}\right)=-\frac{n+1/2}{n}\frac{\cos(n+1/2)t}{2\sin t/2}+\frac{\cos t/2 \sin(n+1/2)t}{4n(\sin t/2)^2}.$$

We put $n+1/2=m$, then

$$\int_0^{\pi}\psi_x(t) g(n, t)\, dt=\int_0^{\pi}\psi_x(t)\left\{-\frac{m}{n}\frac{\cos mt}{2\sin t/2}+\frac{\cos t/2 \sin mt}{4n(\sin t/2)^2}\right\}dt$$
$$=\left(\int_0^{\pi/m}+\int_{\pi/m}^{\pi}\right)\psi_x(t)\left(\frac{\cos t/2 \sin mt}{nt^2}-\frac{m}{n}\frac{\cos mt}{t}\right)dt+o(1)=I+J+o(1).$$

Now
$$I=\int_0^{\pi/m}\psi_x(t)\left\{\frac{\cos t/2\sin mt}{nt^2}-\frac{m}{n}\frac{\cos mt}{t}\right\}dt$$
$$=\frac{1}{n}\int_0^{\pi/m}\psi_x(t)\frac{\cos t/2\sin mt-mt\cos mt}{t^2}dt,$$

and then by integration by parts we get

$$I=\frac{1}{n}\left[\varPsi_x(t)\frac{\cos t/2\sin mt-mt\cos mt}{t^2}\right]_0^{\pi/m}$$
$$-\frac{1}{n}\int_0^{\pi/m}\varPsi_x(t)\frac{d}{dt}\left(\frac{\cos t/2\sin mt-mt\cos mt}{t^2}\right)dt$$

$$=\frac{1}{n}\left[\Psi_x(t)\frac{\cos t/2\sin mt-mt\cos mt}{t^2}\right]_0^{\pi/m}$$
$$+\frac{2}{n}\int_0^{\pi/m}\Psi_x(t)\frac{\cos t/2\sin mt-mt\cos mt}{t^3}dt$$
$$+\frac{1}{n}\int_0^{\pi/m}\Psi_x(t)\frac{-m\cos mt\cos t/2+\frac{1}{2}\sin t/2\sin mt-m\cos mt-m^2t\sin mt}{t^2}dt$$
$$=I_1+I_2+I_3.$$

By the condition (1) we get

$$I_1=O\left(\left[\frac{1}{n}|\Psi_x(t)|\frac{m^3t^3}{t^2}\right]_0^{\pi/m}\right)=o\left(\left[\frac{m^2}{n}t^2\left(\log\frac{1}{t}\right)^\alpha\right]_0^{\pi/m}\right)=o((\log n)^\alpha),$$
$$I_2=O\left(\frac{m^3}{n}\int_0^{\pi/m}|\Psi_x(t)|\,dt\right)=o\left(\frac{m^3}{n}\int_0^{\pi/m}t\left(\log\frac{1}{t}\right)^\alpha dt\right)=o((\log n)^\alpha),$$
$$I_3=\frac{2}{n}\int_0^{\pi/m}\Psi_x(t)\frac{\cos t/2\sin mt-mt\cos mt}{t^3}dt=O\left(\frac{m^3}{n}\int_0^{\pi/m}|\Psi_x(t)|\,dt\right)$$
$$=o\left(\frac{m^3}{n}\int_0^{\pi/m}t\left(\log\frac{1}{t}\right)^\alpha dt\right)=o\left(\frac{m^3}{n}\frac{\pi^2}{m^2}(\log m)^\alpha\right)=o((\log n)^\alpha).$$

On the other hand

$$J=\frac{1}{n}\int_{\pi/m}^{\pi}\psi_x(t)\frac{\cos t/2\sin mt}{t^2}dt-\frac{m}{n}\int_{\pi/m}^{\pi}\psi_x(t)\frac{\cos mt}{t}dt=J_1-J_2,$$

where

$$J_1=\frac{1}{n}\sum_{k=1}^{m-1}(-1)^k\int_0^{\pi/m}\psi_x(t+k\pi/m)\frac{\cos(t+k\pi/m)/2\sin mt}{(t+k\pi/m)^2}dt+o(1)$$
$$=-\frac{1}{n}\sum_{k=1}^{[(m-1)/2]}\int_0^{\pi/m}\left[\psi_x(t+2k\pi/m)\frac{\cos(t+2k\pi/m)/2\sin mt}{(t+2k\pi/m)^2}\right.$$
$$\left.-\psi_x(t+(2k+1)\pi/m)\frac{\cos(t+(2k+1)\pi/m)/2\sin mt}{(t+(2k+1)\pi/m)^2}\right]dt+o(1)$$
$$=-\frac{1}{n}\sum_{k=1}^{[(m-1)/2]}\left[\int_0^{\pi/m}\{\psi_x(t+2k\pi/m)-\psi_x(t+(2k+1)\pi/m)\}\right.$$
$$\cdot\frac{\cos(t+2k\pi/m)/2}{(t+2k\pi/m)^2}\sin mt\,dt$$
$$+\int_0^{\pi/m}\psi_x(t+(2k+1)\pi/m)\left\{\frac{\cos(t+2k\pi/m)/2}{(t+2k\pi/m)^2}\right.$$
$$\left.-\frac{\cos(t+(2k+1)\pi/m)/2}{(t+(2k+1)\pi/m)^2}\right\}\sin mt\,dt+o(1)=J_{11}+J_{12}+o(1),$$

and

$$J_2=\frac{m}{n}\sum_{k=1}^{m-1}(-1)^{k+1}\int_0^{\pi/m}\psi_x(t+k\pi/m)\frac{\cos mt}{t+k\pi/m}dt+o(1)$$
$$=\frac{m}{n}\sum_{k=1}^{[(m-1)/2]}\int_0^{\pi/m}\left[\psi_x(t+2k\pi/m)\frac{\cos mt}{t+2k\pi/m}\right.$$
$$\left.-\psi_x(t+(2k+1)\pi/m)\frac{\cos mt}{t+(2k+1)\pi/m}\right]dt+o(1)$$
$$=\frac{m}{n}\sum_{k=1}^{[(m-1)/2]}\left[\int_0^{\pi/m}\{\psi_x(t+2k\pi/m)-\psi_x(t+(2k+1)\pi/m)\}\frac{\cos mt}{t+2k\pi/m}dt\right.$$

$$+\int_0^{\pi/m}\psi_x(t+(2k+1)\pi/m)\Big(\frac{1}{t+2k\pi/m}-\frac{1}{t+(2k+1)\pi/m}\Big)\cos mt\,dt\Big]+o(1)$$
$$=J_{21}+J_{22}+o(1).$$

Each of J_{ij} $(i,j=1,2)$ is of order $o((\log n)^\alpha)$, as may be seen from [5]. Thus the theorem is proved.

3. Proof of Theorem 5. Let us estimate the order of

$$\frac{1}{A_n^r}\sum_{\nu=1}^{n}A_{n-\nu}^{r-1}\nu B_\nu(x)-\frac{l}{\pi}=\frac{1}{\pi}\int_0^{\pi}\psi_x(t)\Big(\frac{1}{A_n^r}\sum_{\nu=1}^{n}A_{n-\nu}^{r-1}\nu\sin\nu t\Big)dt+o(1)$$
$$=\frac{1}{\pi}\int_0^{\pi}\psi_x(t)(K_n^r(t))'\,dt+o(1),$$

where
$$K_n^r(t)=\frac{1}{A_n^r}\sum_{\nu=0}^{n}A_{n-\nu}^{r-1}\cos\nu t$$
$$=\frac{\cos((n+1/2)t-r\pi/2)}{A_n^r(2\sin t/2)^r}-\frac{A_{n+1}^{r-1}\cos t/2}{A_n^r 2\sin t/2}-\frac{A_{n+2}^{r-2}}{A_n^r}\frac{\cos t}{(2\sin t/2)^2}$$
$$-\frac{1}{A_n^r}\frac{1}{(2\sin t/2)^2}\sum_{\mu=n+3}^{\infty}A_\mu^{r-3}\cos(\mu-n-1)t.$$

Then

$$(K_n^r(t))'=-(n+1/2)\frac{\sin((n+1/2)t-r\pi/2)}{A_n^r(2\sin t/2)^r}-r\frac{\cos((n+1/2)t-r\pi/2)\cos t/2}{A_n^r(2\sin t/2)^{r+1}}$$
$$+\frac{A_{n+1}^{r-1}}{2A_n^r}+\frac{A_{n+1}^{r-1}}{8A_n^r}\cot^2\frac{t}{2}+\frac{A_{n+2}^{r-2}}{A_n^r}\frac{\sin t}{(2\sin t/2)^2}+\frac{A_{n+2}^{r-2}}{A_n^r}\frac{\cos t\cos t/2}{(2\sin t/2)^3}$$
$$+\frac{\cos t/2}{A_n^r(2\sin t/2)^3}\sum_{\mu=n+3}^{\infty}A_\mu^{r-3}\cos(\mu-n-1)t$$
$$+\frac{1}{A_n^r(2\sin t/2)^2}\sum_{\mu=n+3}^{\infty}A_\mu^{r-3}(\mu-n-1)\sin(\mu-n-1)t=\sum_{j=1}^{8}L_j(t).$$

We have, setting $m=n+1/2$,

$$n^{1-r}\int_{\pi/m}^{\pi}\psi_x(t)\frac{\sin(mt-r\pi/2)}{t^r}dt$$
$$=n^{1-r}\int_0^{\pi/m}\sum_{k=1}^{[n/2]}\frac{\psi_x(t+(2k-1)\pi/m)-\psi_x(t+2k\pi/m)}{(t+(2k-1)\pi/m)^r}\sin(mt-r\pi/2)\,dt$$
$$+n^{1-r}\int_0^{\pi/m}\sum_{k=1}^{[n/2]}\psi_x(t+2k\pi/m)\Big(\frac{1}{(t+(2k-1)\pi/m)^r}$$
$$-\frac{1}{(t+2k\pi/m)^r}\Big)\sin(mt-r\pi/2)\,dt+o(1)=I_1+J_1+o(1).$$

Then

$$I_1=n(1+o(1))\sum_{k=1}^{n}\frac{1}{k^r}\int_{\xi_k}^{\eta_k}\{\psi_x(t+(2k-1)\pi/m)-\psi_x(t+2k\pi/m)\}\,dt$$
$$=o\Big(n\sum_{k=1}^{n}\frac{1}{k^r}\frac{(\log n)^\alpha}{n^{2-r}}\Big)=o((\log n)^\alpha),$$

$$J_1=n^{1-r}\sum_{k=1}^{[n/2]}\int_0^{\pi/m}(\psi_x(t+2k\pi/m)-\psi_x(t+(2k+2)\pi/m))$$

$$\cdot\sum_{j=k}^{n}\left(\frac{1}{(t+(2j-1)\pi/m)^r}-\frac{1}{(t+2j\pi/m)^r}\right)\sin(mt-r\pi/2)\,dt$$

$$+n^{1-r}\int_0^{\pi/m}\psi_x(t+2\pi/m)\sum_{j=1}^{n}\left(\frac{1}{(t+(2j-1)\pi/m)^r}\right.$$

$$\left.-\frac{1}{(t+2j\pi/m)^r}\right)\sin(mt-r\pi/2)\,dt=J_{11}+J_{12},$$

where $J_{11}=o\left(n^{1-r}\sum_{k=1}^{n}\sum_{j=k}^{n}\frac{m^r}{j^{r+1}}\frac{(\log n)^\alpha}{n^{2-r}}\right)=o\left(n^{r-1}\sum_{k=1}^{n}\frac{1}{k^r}(\log n)^\alpha\right)$

$$=o((\log n)^\alpha)$$

and $J_{12}=o\left(n^{1-r}\sum_{j=1}^{n}\frac{m^r}{j^{r+1}}\frac{1}{n}(\log n)^\alpha\right)=o((\log n)^\alpha).$

Thus we get $\int_{\pi/m}^{\pi}\psi_x(t)L_1(t)\,dt=o((\log n)^\alpha).$

Secondly, $\frac{1}{n^r}\int_{\pi/m}^{\pi}\frac{\psi_x(t)}{t^{1+r}}\cos(mt-r\pi/2)\,dt$

$$=\frac{1}{n^r}\int_0^{\pi/m}\sum_{k=1}^{[n/2]}\frac{\psi_x(t+(2k-1)\pi/m)-\psi_x(t+2k\pi/m)}{(t+(2k-1)\pi/m)^{r+1}}\cos(mt-r\pi/2)\,dt$$

$$+\frac{1}{n^r}\int_0^{\pi/m}\sum_{k=1}^{[n/2]}\psi_x(t+2k\pi/m)\left(\frac{1}{(t+(2k-1)\pi/m)^{1+r}}\right.$$

$$\left.-\frac{1}{(t+2k\pi/m)^{1+r}}\right)\cos(mt-r\pi/2)\,dt,$$

which is $o((\log n)^\alpha/n^{1-\alpha})+o((\log n)^\alpha)=o((\log n)^\alpha)$. Thus

$$\int_{\pi/m}^{\pi}\psi_x(t)L_2(t)\,dt=o((\log n)^\alpha).$$

Thirdly $\frac{1}{n}\int_{\pi/m}^{\pi}\frac{\psi_x(t)}{t^2}\,dt=\frac{1}{n}\left[\frac{\Psi_x(t)}{t^2}\right]_{\pi/m}^{\pi}+\frac{2}{n}\int_{\pi/m}^{\pi}\frac{\Psi_x(t)}{t^3}dt$

$$=o((\log n)^\alpha)+o\left(\frac{2}{n}\int_{\pi/m}^{\pi}\frac{(\log 1/t)^\alpha}{t^2}\,dt\right)=o((\log n)^\alpha),$$

and hence $\int_{\pi/m}^{\pi}\psi_x(t)L_4(t)\,dt=o((\log n)^\alpha).$

On the other hand

$$\int_{\pi/m}^{\pi}\frac{\psi_x(t)}{t^3}\cos(\mu-n-1)t\,dt$$

$$=\left[\frac{\Psi_x(t)}{t^3}\cos(\mu-n-1)t\right]_{\pi/m}^{\pi}+\int_{\pi/m}^{\pi}\Psi_x(t)\left(\frac{3\cos(\mu-n-1)t}{t^4}\right.$$

$$\left.-\frac{(\mu-n-1)\sin(\mu-n-1)t}{t^3}\right)dt$$

$$=o(n^2(\log n)^\alpha)+o\left(\int_{\pi/m}^{\pi}\left(\frac{(\log 1/t)^\alpha}{t^3}+\frac{\mu(\log 1/t)^\alpha}{t^2}\right)dt\right)$$

$$=o(n^2(\log n)^\alpha)+o(n^2(\log n)^\alpha+\mu n(\log n)^\alpha),$$

hence we get $$\int_{\pi/m}^{\pi}\psi_x(t)L_7(t)\,dt$$

$$=o\left(n^{2-r}(\log n)^\alpha\sum_{\mu=n+1}^{\infty}\frac{1}{\mu^{3-r}}\right)+o\left(n^{1-r}(\log n)^\alpha\sum_{\mu=n+1}^{\infty}\frac{1}{\mu^{2-r}}\right)=o((\log n)^\alpha).$$

Furthermore

$$\int_{\pi/m}^{\pi}\psi_x(t)L_8(t)dt=\frac{1}{A_n^r}\int_{\pi/m}^{\pi}\frac{\psi_x(t)}{(2\sin t/2)^2}\left(\sum_{\mu=n+3}^{\infty}A_\mu^{r-3}(\mu-n-1)\sin(\mu-n-1)t\right)dt,$$

where

$$\sum_{\mu=n+3}^{\infty}A_\mu^{r-3}(\mu-n-1)\sin(\mu-n-1)t$$

$$=\frac{d}{dt}\left(\sum_{\mu=n+3}^{\infty}A_\mu^{r-3}\cos(\mu-n-1)t\right)=\frac{d}{dt}\left(\mathcal{R}\sum_{\mu=n+3}^{\infty}A_\mu^{r-3}e^{i(\mu-n-1)t}\right)$$

$$=\frac{d}{dt}\left(\mathcal{R}\left(e^{-i(n+1)t}\sum_{\mu=n+3}^{\infty}A_\mu^{r-3}\right)\right)=\frac{d}{dt}(\mathcal{R}(e^{-i(n+1)t}A_{n+2}^{r-2}))$$

$$=-A_{n+2}^{r-2}(n+1)\sin(n+1)t.$$

Hence $$\int_{\pi/m}^{\pi}\psi_x(t)L_8(t)\,dt=\frac{A_{n+2}^{r-2}}{A_n^r}(n+1)\int_{\pi/m}^{\pi}\frac{\psi_x(t)}{(2\sin t/2)^2}\sin(n+1)t\,dt$$

which is $o((\log n)^\alpha)$ by the estimation similar to J_1. The remaining integral is also $o((\log n)^\alpha)$.

It remains to estimate

$$\int_0^{\pi/m}\psi_x(t)(K_n^r(t))'\,dt=\frac{1}{A_n^r}\sum_{\nu=0}^{n}A_{n-\nu}^{r-1}\nu\int_0^{\pi/m}\psi_x(t)\cos\nu t\,dt$$

$$=\frac{1}{A_n^r}\sum_{\nu=0}^{n}A_{n-\nu}^{r-1}\,o\left(\frac{\nu}{m}(\log m)^\alpha\right)=o((\log n)^\alpha).$$

Thus the theorem is proved.

References

[1] L. Fejér: Über die Bestimmung des Sprunges der Funktion aus ihrer Fourierreihe, Jour. f. Math., **142** (1913).

[2] B. Singh: On a sequence of Fourier coefficients, Proc. Amer. Math. Soc., **8** (1957).

[3] O. Szász: The jump of a function determined by its Fourier coefficients, Duke Math. Jour., **4** (1938).

[4] ——: The generalized jump of a function and Gibbs' phenomenon, Duke Math. Jour., **11** (1944).

[5] M. Satô: Fourier series. XVII. Order of partial sums and convergence theorem, Proc. Japan Acad., **33** (1957).

[6] A. Zygmund: Trigonometrical series, Warszawa (1935).