128. On Non-linear Partial Differential Equations of Parabolic Types. I

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Introduction. In this paper we shall consider the following non-linear partial differential equations of parabolic types:

$$\partial^2 u/\partial x^2 - \partial u/\partial y = f(x, y, u)$$
$$\partial^2 u/\partial x^2 - \partial u/\partial y = f(x, y, u, \partial_x u),$$
$$\partial^2 u/\partial x^2 - \partial u/\partial y = f(x, y, u, \partial_x u, \partial_y u).$$ [1]

Our main aim is to solve the first boundary value problem of the first equation by so-called Perron's method which was originally used by O. Perron to solve the Dirichlet problem for Laplace's equation[2] and later used by W. Sternberg for the equation of heat conduction.[3] Recently, Prof. T. Satō modified this method and solved the Dirichlet problem for the non-linear equation of elliptic type.[4] In his papers, however, as an inevitable consequence of the method used there and of the non-linearity of the equation, he had to extend the meaning of the Laplacian operator. To solve our problem following Satō's idea, we must also extend the parabolic differential operator $\partial^2/\partial x^2 - \partial/\partial y$ to a generalized heat operator $\square$. This generalization is shown in §1. Thus, the equations considered in this paper are of the following types:

$$(\mathrm{E}_1) \qquad \square u = f(x, y, u),$$
$$(\mathrm{E}_2) \qquad \square u = f(x, y, u, \partial_x u),$$
$$(\mathrm{E}_3) \qquad \square u = f(x, y, u, \partial_x u, \partial_y u).$$

In §1, after the definition of generalized heat operator $\square$, we give some notations and definitions needed in the sequel. In §2 we state and prove some comparison theorems. Theorem 2.7 and its corollaries play important roles later. In §3 we give a uniqueness condition. In §4 we give some existence theorems which show the existence of solutions under some restricted conditions.[5] Harnack's first

1) We use the notations $\partial_x u$ and $\partial_y u$ for $\partial u/\partial x$ and $\partial u/\partial y$ respectively.

2) O. Perron: Eine neue Behandlung der ersten Randwertaufgaben für $\triangle u=0$, Math. Zeitschr., **18** (1923).

3) W. Sternberg: Ueber die Gleichung der Wärmeleitung, Math. Ann., **101** (1929).

4) T. Satō: Sur l'équations aux dérivées partielles $\triangle z=f(x, y, z, p, q)$, Comp. Math., **12** (1954) and Sur l'équation aux dérivées partielles $\triangle z=f(x, y, z, p, q)$ II (to appear). See also M. Hukuhara and T. Satō: Theory of Differential Equations (in Japanese), Kyōritu Publ. Co. Ltd., Tokyo (1957), cited as Hukuhara-Satō.

5) In his paper which was sent to Prof. T. Satō recently, Prof. B. Pini also proves similar theorems in §§2, 3 and 4 of the present paper independently. Therefore we shall omit the details of the proofs there. See B. Pini: Sul primo problema di valori al contorno per l'equazione parabolica non lineare del secondo ordine, Rend. del Sem. Mat. d. Università di Padova (1957), cited as B. Pini [1].

and second theorems about harmonic functions are extended to our case in §5. We introduce quasi-superior and quasi-inferior function in §6, Φ_β- and Ψ_β-functions in §7. After giving a global existence theorem in §8, barriers are defined in §9. §10 gives the following fundamental result: under some conditions about $f(x, y, u)$ the equation (E_1) is always solvable for the domain on which the equation of heat conduction is solvable. In §11 we extend our results to higher dimensional spaces.

In terminating the Introduction I am deeply grateful for this opportunity of thanking Professor Tokui Satō who drew my attention to problems treated in the present paper, encouraged me in innumerable discussions and gave me many criticisms and improvements during the preparation of this paper.

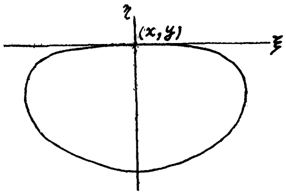
1. Preliminaries. GENERALIZED HEAT OPERATOR

Let $u(x, y)$ be a function defined and continuous around $P(x, y)$. We define

$$\overline{\square} u(x, y)=\overline{\lim_{r\to+0}}\frac{3\sqrt{3}}{\sqrt{2\pi}r^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{u(x+\sqrt{2}\,r\sin\theta\sqrt{\log\operatorname{cosec}^2\theta},\ y-r^2\sin^2\theta)$$
$$-u(x, y)\}\cos\theta\sqrt{\log\operatorname{cosec}^2\theta}\,d\theta,$$

$$\underline{\square} u(x, y)=\underline{\lim_{r\to+0}}\frac{3\sqrt{3}}{\sqrt{2\pi}r^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{u(x+\sqrt{2}\,r\sin\theta\sqrt{\log\operatorname{cosec}^2\theta},\ y-r^2\sin^2\theta)$$
$$-u(x, y)\}\cos\theta\sqrt{\log\operatorname{cosec}^2\theta}\,d\theta.$$

If $\overline{\square} u(x, y)$ and $\underline{\square} u(x, y)$ coincide, it is denoted by $\square u(x, y)$.



$$\begin{cases}\xi=x+\sqrt{2}r\sin\theta\sqrt{\log\operatorname{cosec}^2\theta}\\ \eta=y-r^2\sin^2\theta\end{cases}$$

This operator $\square$ has the following properties:

i)[6] *If $u(x, y)$ belongs to the class C^2 with respect to x and to the class C^1 with respect to y, then*

$$\square u(x, y)=\partial^2 u/\partial x^2-\partial u/\partial y.$$

ii) *If $u_n(x, y)$ converge uniformly to $u(x, y)$ in a domain D and $\overline{\square} u_n(x, y)$, $\underline{\square} u_n(x, y)$ also converge uniformly in D to the same function $\tilde{u}(x, y)$, then*

$$\square u(x, y)=\lim_{n\to\infty}\overline{\square} u_n(x, y)=\lim_{n\to\infty}\underline{\square} u_n(x, y)=\tilde{u}(x, y).$$

iii) *If $g(x, y)$ is continuous and integrable in a domain D, then*

$$\square\left(-\frac{1}{2\sqrt{\pi}}\iint_D U(x, y; \xi, \eta)g(\xi, \eta)\,d\xi\,d\eta\right)=g(x, y), \qquad (x, y)\in D,$$

6) The definition of $\square$ and the property i) are due to B. Pini. See B. Pini [1] and his another paper: Maggioranti e minoranti delle solzioni delle equazioni paraboliche, Ann. di Mat., **37** (1954).

where

$$U(x, y; \xi, \eta)=\begin{cases} \dfrac{1}{y-\eta}\exp\left(-\dfrac{(x-\xi)^2}{4(y-\eta)}\right) & \eta<y \\ 0 & \eta\geq y. \end{cases}$$

We call the operator $\Box$ *generalized heat operator.*

NOTATIONS. Let M be a set on (x, y)-plane. We use following notations:

Diameter of M: $d=d(M)=\sup\{\operatorname{dist}(P, Q);\ P, Q\in M\}$.

Height of M: $h=h(M)=\sup\{y-y';\ (x, y), (x', y')\in M\}$.

Width of M: $w=w(M)=\sup\{x-x';\ (x, y), (x', y')\in M\}$.

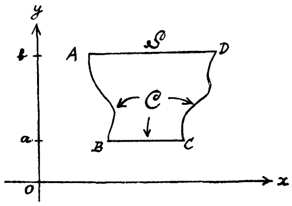
$\{(x, y);\ (x, y)\in M,\ y<y_0\}$ is denoted by M_{y_0}.

Let $f(x, y)$ be a function defined on a set M. We define

$$\bar{f}(x, y)=\overline{\lim_{(\xi,\eta)\to(x,y)}} f(\xi, \eta),\quad \underline{f}(x, y)=\underline{\lim_{(\xi,\eta)\to(x,y)}} f(\xi, \eta)$$

where $(x, y)\in M'$ and $(\xi, \eta)\in M$.

DEFINITIONS AND NOTATIONS OF *p-domain* AND C^1-*p-domain.* A point set in (x, y)-plane is called *p-domain* if its boundary consists of the following four parts: upper bounding segment, say AD on the straight line $y=b$, lower bounding segment, say BC on the straight line $y=a$ (B and C may coincide), and two continuons curves connecting A to B and D to C, to which we assume that these curves are representable as $x=\lambda_1(y)$ and $x=\lambda_2(y)$ where λ_1 and λ_2 are one valued continuous functions on $a\leq y\leq b$ and moreover $\lambda_1(y)<\lambda_2(y)$ on $a<y\leq b$. We call these two curves *side curves* of the p-domain. If D is such a p-domain, we denote always the upper bounding segment by $\mathcal{S}$ (not including its two end points), and both side curves together with the lower bounding segment (including their end points) by $\mathcal{C}$. Moreover, we denote the interior of D by $(\mathcal{C}, \mathcal{S})$. We denote $(\mathcal{C}, \mathcal{S})\smile\mathcal{S}$, $(\mathcal{C}, \mathcal{S})\smile\mathcal{C}$ and $(\mathcal{C}, \mathcal{S})\smile\mathcal{C}\smile\mathcal{S}$ by $(\mathcal{C}, \mathcal{S}]$, $[\mathcal{C}, \mathcal{S})$ and $[\mathcal{C}, \mathcal{S}]$ respectively. In this paper, we use the term "*p-domain* $(\mathcal{C}, \mathcal{S}]$" or "*p-domain* $[\mathcal{C}, \mathcal{S}]$", etc, so that the word "*p-domain*" does not always mean an open set.



If the both side curves belong to the class C^1 and $\lambda_1(a)<\lambda_2(a)$, we call the p-domain C^1-*p-domain* and in this case we use always $\mathcal{L}$ instead of $\mathcal{C}$.

DEFINITIONS OF SOLUTIONS. Whenever we speak of solutions on $(\mathcal{C}, \mathcal{S}]$ of the equations (E_1), (E_2) and (E_3) respectively, we assume always that they are continuous functions satisfying these equations in $(\mathcal{C}, \mathcal{S}]$; and moreover, for the solution of (E_2) or (E_3) we assume the existence of their partial derivatives appeared in the right hand sides of the equations respectively.

DEFINITION. Suppose that $f(x, y)$ is a function defined on the set $E_1\times E_2$. We say that $f(x, y)$ is *quasi-bounded with respect to* y if

$f(x, y)$ is bounded on $E_1 \times K$, where K is any compact set in E_2.

2. Comparison theorems. We begin by proving,

THEOREM 2.1.[7] *Let $f(x, y, u, p, q)$ be a function defined on $(x, y) \in (\mathcal{C}, \mathcal{S}]$ and $(u, p, q) \in \mathcal{E}$,[8] and let $\omega(x, y)$ be a function which is continuous and has $\partial_x\omega(x, y)$ and $\partial_y\omega(x, y)$ on $(\mathcal{C}, \mathcal{S}]$. Suppose that we have*

$$\underline{\square}\omega(x, y) < f(x, y, u, \partial_x\omega(x, y), \partial_y\omega(x, y)) \tag{2.1}$$

for $(x, y) \in (\mathcal{C}, \mathcal{S})$, $\omega(x, y) < u$, $(u, \partial_x\omega(x, y), \partial_y\omega(x, y)) \in \mathcal{E}$ and

$$\underline{\square}\omega(x, y) < f(x, y, u, \partial_x\omega(x, y), q) \tag{2.1'}$$

for $(x, y) \in \mathcal{S}$, $\omega(x, y) < u$, $\partial_y\omega(x, y) \leq q$, $(u, \partial_x\omega(x, y), q) \in \mathcal{E}$. If, for $(x, y) \in (\mathcal{C}, \mathcal{S}]$ and $(x_0, y_0) \in \mathcal{C}$,

$$\varliminf_{(x,y)\to(x_0,y_0)} (\omega(x, y) - u(x, y)) \geq 0, \tag{2.2}$$

then

$$\omega(x, y) \geq u(x, y) \tag{2.3}$$

on $(\mathcal{C}, \mathcal{S}]$, where $u(x, y)$ is a solution of (E_3) on $(\mathcal{C}, \mathcal{S}]$.

PROOF. For any $\varepsilon > 0$ it follows from (2.2) that there is a neighbourhood U of $\mathcal{C}$ such that $\omega(x, y) - u(x, y) > -\varepsilon$ for $(x, y) \in (\mathcal{C}, \mathcal{S}] \frown U$. If there is a point (x, y) such that $\omega(x, y) - u(x, y) \leq -\varepsilon$ in $(\mathcal{C}, \mathcal{S}] - U$, $\omega(x, y) - u(x, y)$ attains its minimum at the point (x_1, y_1) of $(\mathcal{C}, \mathcal{S}] - U$. If $(x_1, y_1) \in (\mathcal{C}, \mathcal{S})$, then $\omega(x_1, y_1) < u(x_1, y_1)$, $\partial_x\omega(x_1, y_1) = \partial_x u(x_1, y_1)$, $\partial_y\omega(x_1, y_1) = \partial_y u(x_1, y_1)$, $\underline{\square}\omega(x_1, y_1) \geq \square u(x_1, y_1)$. Therefore we have $\underline{\square}\omega(x_1, y_1) \geq f(x_1, y_1, u(x_1, y_1), \partial_x\omega(x_1, y_1), \partial_y\omega(x_1, y_1))$, which contradicts (2.1). If $(x_1, y_1) \in \mathcal{S}$, then $\omega(x_1, y_1) < u(x_1, y_1)$, $\partial_x\omega(x_1, y_1) = \partial_x u(x_1, y_1)$, $\partial_y\omega(x_1, y_1) \leq \partial_y u(x_1, y_1)$, $\underline{\square}\omega(x_1, y_1) \geq \square u(x_1, y_1)$. Therefore we have $\underline{\square}\omega(x_1, y_1) \geq f(x_1, y_1, u(x_1, y_1), \partial_x\omega(x_1, y_1), \partial_y u(x_1, y_1))$, which contradicts (2.1'). Q.E.D.

THEOREM 2.1$^{\text{bis}}$. *Let $f(x, y, u, p)$ be a function defined on $(x, y) \in (\mathcal{C}, \mathcal{S}]$ and $(u, p) \in \mathcal{E}_1$,[9] and let $\omega(x, y)$ be a function which is defined and differentiable with respect to x on $(\mathcal{C}, \mathcal{S}]$. Suppose that*

$$\underline{\square}\omega(x, y) < f(x, y, u, \partial_x\omega(x, y))$$

for $(x, y) \in (\mathcal{C}, \mathcal{S}]$, $\omega(x, y) < u$, $(u, \partial_x\omega(x, y)) \in \mathcal{E}_1$. Then, (2.2) implies (2.3), where $u(x, y)$ is a solution of (E_2) on $(\mathcal{C}, \mathcal{S}]$.

THEOREM 2.2. *Under the same assumptions of Theorem 2.1 or Theorem 2.1$^{\text{bis}}$,*

$$\varliminf_{(x,y)\to(x_0,y_0)} (\omega(x, y) - u(x, y)) > 0, \quad (x, y) \in (\mathcal{C}, \mathcal{S}], (x_0, y_0) \in \mathcal{C}, \tag{2.4}$$

implies $\omega(x, y) > u(x, y)$ on $(\mathcal{C}, \mathcal{S}]$.

THEOREM 2.3.[10] *Let $f_1(x, y, u, p)$ and $f_2(x, y, u, p)$ be functions defined for $(x, y) \in (\mathcal{C}, \mathcal{S}]$ and $-\infty < u, p < +\infty$. Suppose that*

$$f_1(x, y, u_1, p) < f_2(x, y, u_2, p) \tag{2.5}$$

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7) This theorem is due to T. Satō, see Hukuhara-Satō, pp. 286-287.

8) $\mathcal{E}$ is a set in (u, p, q)-space.

9) $\mathcal{E}_1$ is a set in (u, p)-space.

for $(x,y)\in(C,S]$, $u_1<u_2$. *If* $u_1(x,y)$ *and* $u_2(x,y)$ *are continuous functions on* $(C,S]$ *which are differentiable with respect to* x *and satisfy*

$$\overline{\Box}u_1(x,y)\leq f_1(x,y,u_1(x,y),\ \partial_x u_1(x,y))$$

and

$$\underline{\Box}u_2(x,y)\geq f_2(x,y,u_2(x,y),\ \partial_x u_2(x,y))$$

respectively, then $\underline{\lim}\,\{u_1(x,y)-u_2(x,y)\}\geq 0$ *on* C *implies* $u_1(x,y)\geq u_2(x,y)$ *on* $(C,S]$.

PROOF. By the assumption that $\underline{\lim}\,\{u_1(x,y)-u_2(x,y)\}\geq 0$ on C, for any $\varepsilon>0$ we can find a neighbourhood U of C such that $u_1(x,y)-u_2(x,y)>-\varepsilon$ on $(C,S]\frown U$. If the set of points such that $u_1(x,y)-u_2(x,y)\leq-\varepsilon$ is not vacuous in $(C,S]-U$, there is a point (x_0,y_0) in $(C,S]$ such that $u_1(x,y)-u_2(x,y)$ attains its minimum at that point. At the point (x_0,y_0) we have $u_1(x_0,y_0)<u_2(x_0,y_0)$ and $\partial_x u_1(x_0,y_0)=\partial_x u_2(x_0,y_0)$. Then by (2.5) we have $\overline{\Box}(u_1(x_0,y_0)-u_2(x_0,y_0))\leq\overline{\Box}\,u_1(x_0,y_0)-\underline{\Box}\,u_2(x_0,y_0)\leq f_1(x_0,y_0,u_1(x_0,y_0),\ \partial_x u_1(x_0,y_0))-f_2(x_0,y_0,u_2(x_0,y_0),\ \partial_x u_2(x_0,y_0))<0$. On the other hand, since $u_1(x,y)-u_2(x,y)$ attains its minimum at (x_0,y_0) we have $\overline{\Box}\{u_1(x_0,y_0)-u_2(x_0,y_0)\}\geq 0$. These two inequalities contradict each other. Q.E.D.

THEOREM 2.4. *Under the same assumptions of Theorem 2.3, if* $\underline{\lim}\,\{u_1(x,y)-u_2(x,y)\}>0$ *on* C, *then* $u_1(x,y)>u_2(x,y)$ *on* $(C,S]$.

REMARK. Similar theorems to Theorems 2.1, 2.1^{bis}, 2.2, 2.3 and 2.4 with changing inequality signs hold true.

THEOREM 2.5. *Let* $[C,S]$ *be a p-domain such that there is a solution of* $\Box u=0$ *which is continuous on* $[C,S]$ *and which admits any given continuous boundary value on* C. *Then there is one and only one*[11] *solution of* $\Box u=-1$ *which is continuous on* $[C,S]$ *and vanishes on* C, *and for such solution* $\psi(x,y)$, *we have*

$$0\leq\psi(x,y)\leq d(d+2)/2 \tag{2.6}$$

on $[C,S]$, *where* $d=d(C,S)$.

PROOF. It is easily seen that $\{(x-x_0)^2-2(y-y_0)\}/4$ satisfies $\Box u=1$. Let $\varphi(x,y)$ be a solution of $\Box u=0$ which is continuous on $[C,S]$ and which admits the boundary value $\{(x-x_0)^2-2(y-y_0)\}/4$ on C. Then

$$\psi(x,y)=\varphi(x,y)-[(x-x_0)^2-2(y-y_0)]/4$$

is a solution of $\Box u=-1$ which is continuous on $[C,S]$ and which vanishes on C. Hence $\psi(x,y)\geq 0$ by Theorem 2.1. Since $\varphi(x,y)$ admits its maximum and minimum on C,

$$\psi(x,y)\leq 2\,\mathrm{Max}\,\{[(x-x_0)^2-2(y-y_0)]/4;\ (x,y)\in C\}\leq d(d+2)/2,$$

where $(x_0,y_0)\in(C,S)$. Q.E.D.

THEOREM 2.6. *Let* $\psi(x,y)$ *be the unique solution of* $\Box u=-1$ *which is continuous on* $[\mathcal{L},S]$ *and which vanishes on* $\mathcal{L}$. *Then*

$$0\leq\psi(x,y)\leq 2w(\mathcal{L},S)\sqrt{h(\mathcal{L},S)/\pi} \tag{2.7}$$

on $[\mathcal{L},S]$.

10) B. Pini [1] includes analogous theorems to our Theorems 2.1^{bis}, 2.2 and 2.3.

11) Uniqueness is shown in the next section.

PROOF. Let $G(x, y; \xi, \eta)$ be Green's function for the equation of heat conduction on $[\mathcal{L}, \mathcal{S}]$. Then we have

$$|G(x, y; \xi, \eta)| \leq \frac{1}{\sqrt{\pi(y-\eta)}}, \quad \eta < y.$$

Therefore,

$$\begin{aligned} |\psi(x, y)| &= \left| \iint_{[\mathcal{L}, \mathcal{S}]} G(x, y; \xi, \eta)\, d\xi\, d\eta \right| \\ &\leq \frac{1}{\sqrt{\pi}} \left| \iint_{[\mathcal{L}, \mathcal{S}]_y} \frac{1}{\sqrt{y-\eta}}\, d\xi\, d\eta \right| \\ &\leq \frac{1}{\sqrt{\pi}} 2\sqrt{h(\mathcal{L}, \mathcal{S})}\, w(\mathcal{L}, \mathcal{S}). \end{aligned}$$

Q.E.D.

THEOREM 2.7. *Let $(C, \mathcal{S}]$ be a p-domain which satisfies the condition in Theorem 2.5, and let $f(x, y, u, p)$, $F(x, y, u, p)$ be functions defined for $(x, y) \in (C, \mathcal{S}]$, $-\infty < u, p < +\infty$ and suppose that*

$$f(x, y, u, p) \begin{cases} >0 & u>0 \\ =0 & u=0 \\ <0 & u<0 \end{cases} \tag{2.8}$$

$$|F(x, y, u, p)| \leq M.$$

If $\square u = f(x, y, u, \partial_x u) + F(x, y, u, \partial_x u)$ has a solution $u(x, y)$ which is continuous on $[C, \mathcal{S}]$ and which vanishes on C, then the solution satisfies

$$|u(x, y)| \leq M\psi(x, y) \tag{2.9}$$

on $[C, \mathcal{S}]$, where $\psi(x, y)$ is the function given in Theorem 2.5.

PROOF. Let $v(x, y) = u(x, y) - M'\psi(x, y)$, where M' is any positive constant $> M$. Then, $v(x, y)$ is a solution of

$$\begin{aligned} \square v &= \square u - M' \\ &= f(x, y, u, \partial_x u) + F(x, y, u, \partial_x u) + M' \\ &= f(x, y, v + M'\psi, \partial_x v + M'\partial_x \psi) \\ &\qquad + F(x, y, v + M'\psi, \partial_x v + M'\partial_x \psi) + M'. \end{aligned}$$

Since $\psi \geq 0$, by the assumption (2.8), we have

$f(x, y, v + M'\psi, \partial_x v + M'\partial_x \psi) + F(x, y, v + M'\psi, \partial_x v + M'\partial_x \psi) + M' > 0$ for $v > 0$. Since $v(x, y)$ vanishes on C, by Theorem 2.3 we have $v(x, y) \leq 0$ on $(C, \mathcal{S}]$, i.e.

$$u(x, y) \leq M'\psi(x, y).$$

Similarly, we have $-M'\psi(x, y) \leq u(x, y)$. Thus we have

$$|u(x, y)| \leq M'\psi(x, y).$$

Since M' is any constant greater than M, we have

$$|u(x, y)| \leq M\psi(x, y).$$

Q.E.D.

COROLLARY 1. *Under the same assumptions in Theorem 2.7, we have*

$$|u(x, y)| \leq Md(d+2)/2. \tag{2.10}$$

COROLLARY 2. *Moreover, if the domain is the C^1-p-domain $[\mathcal{L}, \mathcal{S}]$ we have*

$$|u(x, y)| \leq 2Mw(\mathcal{L}, \mathcal{S})\sqrt{h(\mathcal{L}, \mathcal{S})/\pi}. \tag{2.11}$$