126. On Generalized Continuous Convergence

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Recently, H. Schaefer [4] introduced a notion of generalized continuous convergence for a sequence of functions on a topological space. On the other hand, the present writer [1] proved that pseudo-compact completely regular space was characterised by theorems of Ascoli type and Arzela type. In this Note, we shall show that such a space is characterised by a notion of *generalized continuous convergence* (in the sense of H. Schaefer) which is defined below. Let S be a T_2-space, and let $f(x)$ be a real function defined on S. For a point x_0 of S, we shall define the *oscillation* $\sigma(f, x_0)$ of $f(x)$ at the point x_0 as follows:

$$\sigma(f, x_0) = \inf_{U} \sup_{x, x' \in U} |f(x) - f(x')|,$$

where inf. takes all neighbourhood U of x_0.

Let U be a neighbourhood of x_0, and $f_n(x)$ be a sequence of functions defined on S, then we set $S(n, U) = \bigcup_{k \geq n} f_k(U)$. The set $\{S(n, U)\}$ for n and U is *a base of filter on real line*. We denote the filter generated by $\{S(n, U)\}$ by $F(x_0, f_n)$.

Definition 1. *A sequence $\{f_n(x)\}$ of functions is said to be continuously convergent at a point x_0 of S, if the filter $F(x_0, f_n)$ converges on reals* (see H. Schaefer [4, p. 424]).

$f_n(x)$ is said to be *uniformly convergent at a point* $x_0 \in S$, if for a given positive ε, there are a neighbourhood U of x_0 and an integer N such that $|f_m(x) - f_n(x)| < \varepsilon$ for all x of U and $m, n \geq N$.

In his paper [1], the present author has proved that *a completely regular space is pseudo-compact if and only if every sequence of continuous functions which is uniformly convergent at each point is uniformly convergent.*

H. Schaefer [4] has obtained that a sequence $\{f_n(x)\}$ of continuous functions on a topological space converges continuously at a point x_0 if and only if it is uniformly convergent at the point x_0 and $\lim_{n \to \infty} (x_0, f_n) = 0$.

Therefore, if a completely regular space S is pseudo-compact, and a sequence $\{f_n(x)\}$ of continuous functions is continuously convergent at each point, its convergence is uniform on S. Conversely, if every sequence of continuous functions which converges continuously at each point on a completely regular space is uniformly convergent, then it is pseudo-compact. Suppose that it is not pseudo-compact, then there are pairwise disjoint countable non-void open sets $\{O_n\}$ having no cluster point. For every n and a fixed point a of O_n, we shall take a continuous function $f_n(x)$ such that $0 \leq f_n(x) \leq 1$ at every point, $f_n(a_n)=1$ and $f_n(x)=0$ for x of the complement of O_n. For any point x_0, and a neighbourhood U of x_0, since $S(n, U)=\bigcup_{k\geq n} f_k(U)$, the filter $F(x_0, f_n)$ converges to 0. Therefore f_n is continuously convergent in the sense of H. Schaefer, but, $f_n(x)$ is not uniformly convergent, which is a contradiction. Therefore we have the following

Theorem 1. A completely regular space is pseudo-compact, if and only if every sequence of continuous functions which is continuously convergent in the sense of H. Schaefer at each point is uniformly convergent.

In general a continuously convergent sequence in the ordinary sense[1)] does not imply the case of H. Schaefer sense. It is sufficient to consider that a compact space constructed by the product of uncountable infinitely many compact spaces and $f_n(x)=n$ on it.

Following C. Kuratowski, we shall define *strictly continuous convergence* as follows. A sequence $\{f_n(x)\}$ of functions on a topological space is said to be *strictly continuous convergent to* $f(x)$, if the convergence of $\{f(x_n)\}$ implies the convergence of $\{f_n(x_n)\}$ and $\lim f_n(x_n) = \lim f(x_n)$.

In my Note [3], we proved that any weakly separable completely regular space is countably compact, if and only if every sequence of continuous functions which converges continuously to a continuous function is strictly continuous convergent to it. In order to prove a correspondence theorem, we shall use the notion of continuous convergence in the sense of H. Schaefer.

Theorem 2. A completely regular topological space is countably compact, if and only if every sequence of continuous functions which is continuous convergent in the sense of H. Schaefer to a continuous function is strictly continuous convergent to the function.

Proof. Let $f_n(x)$, $f(x)$ be continuous functions on a completely regular space S, and $f_n(x)$ is continuously convergent in the sense of H. Schaefer to $f(x)$. Suppose that $f_n(x)$ is not strictly continuous convergent to $f(x)$. Then there is a sequence $\{x_n\}$ of points of S such that $\{f(x_n)\}$ is convergent to α and $\{f_n(x_n)\}$ is not convergent to α. Therefore we can find a subsequence $\{f_{n_i}(x_{n_i})\}$ such that every subsequence of it does not converge to α. Since S is countably compact, $\{x_{n_i}\}$ has at least one cluster point x_0. On the other hand, by hypothesis, $\mathfrak{F}(x_0, f_{n_i})$ is convergent to $f(x_0)$. This shows that there is a

1) On the ordinary continuous convergence and its related properties, see K. Iséki [2, 3].

subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $f_{n_{i_j}}(x_{n_{i_j}}) \to f(x_0)$. By the continuity of $f(x)$ at the point x_0, x_0 is a cluster point of $\{x_n\}$ and $f(x_n) \to \alpha$, we have $f(x_n) \to f(x_0)$. Hence $\lim f_{n_{i_j}}(x_{n_{i_j}}) = \lim f(x_{n_{i_j}})$, which is a contradiction. Therefore, $f_n(x)$ is strictly continuous convergent to $f(x)$.

To prove the converse, suppose that S is not countably compact, then we can find easily that there are a countable set $\{a_n\}$ and pairwise disjoint countable open sets O_n containing a_n for every n. Therefore, if we take continuous functions $f_n(x)$ such that $0 \leq f_n(x) \leq 1$ on S, $f_n(a_n) = 1$, and $f_n(x) = 0$ on $S - O_n$, $\{f_n(x)\}$ is continuous convergent in the sense of H. Schaefer to 0. On the other hand, $0 = \lim_{n\to\infty} f(a_n) \neq \lim_{n\to\infty} f_n(a_n) = 1$. Hence S is countably compact.

References

[1] K, Iséki: A characterisation of pseudo-compact spaces,[2] Proc. Japan Acad., **33**, 320–322 (1957).

[2] K. Iséki: A theorem on continuous convergence, Proc. Japan Acad., **33**, 355–356 (1957).

[3] K. Iséki: Pseudo-compactness and strictly continuous convergence, Proc. Japan Acad., **33**, 424–428 (1957).

[4] H. Schaefer: Stetige Konvergenz in allgemeinen topologischen Räumen, Archiv der Math., **6**, 423–427 (1955).

2) Theorem 7 in this paper should be read as follows: *Let S be a normal space. S is countably compact, if and only if every sequence of continuous function which is simply-uniformly convergent to a function converges quasi uniform to the function on S.* In my other paper: Pseudo-compactness and μ-convergence, Proc. Japan Acad., **33**, 368–371 (1957), we proved that the condition characterises the pseudo-compactness.